SANT LONGOWAL INSTITUTE OF ENGINEERING AND TECHNOLOGY
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Course Materials

of

Strength of Materials (PCME-523)

Submitted By:

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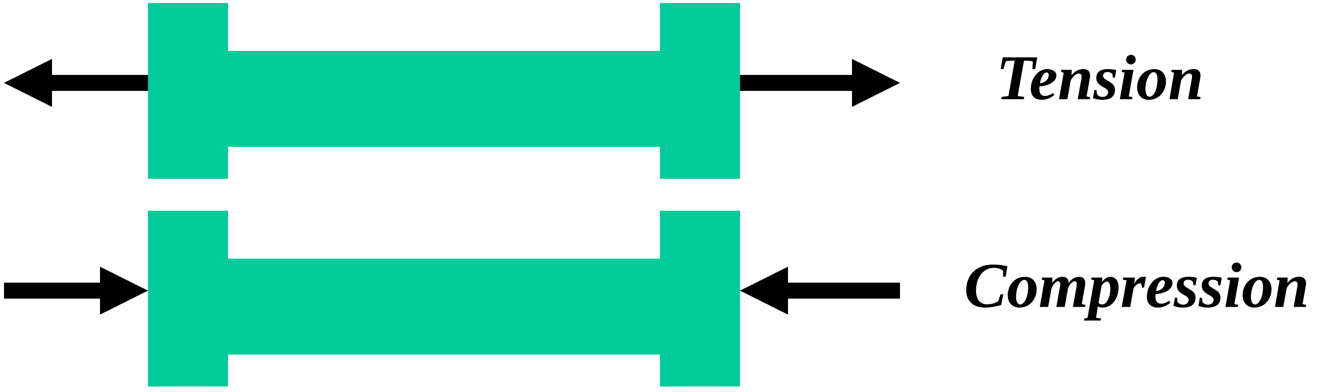
A.P(ME)

Mechanics of Materials

- ✓ The three fundamental areas of engineering mechanics are statics, dynamics, and Mechanics of materials.
- ✓ Statics and dynamics are devoted primarily to the study of the external effects upon rigid bodies—that is, bodies for which the change in shape (deformation) can be neglected.
- ✓ In contrast, Mechanics of materials deals with the internal effects and deformations that are caused by the applied loads.
- ✓ Both considerations are of paramount importance in design.
- ✓ A machine part or structure must be strong enough to carry the applied load without breaking and, at the same time, the deformations must not be excessive.

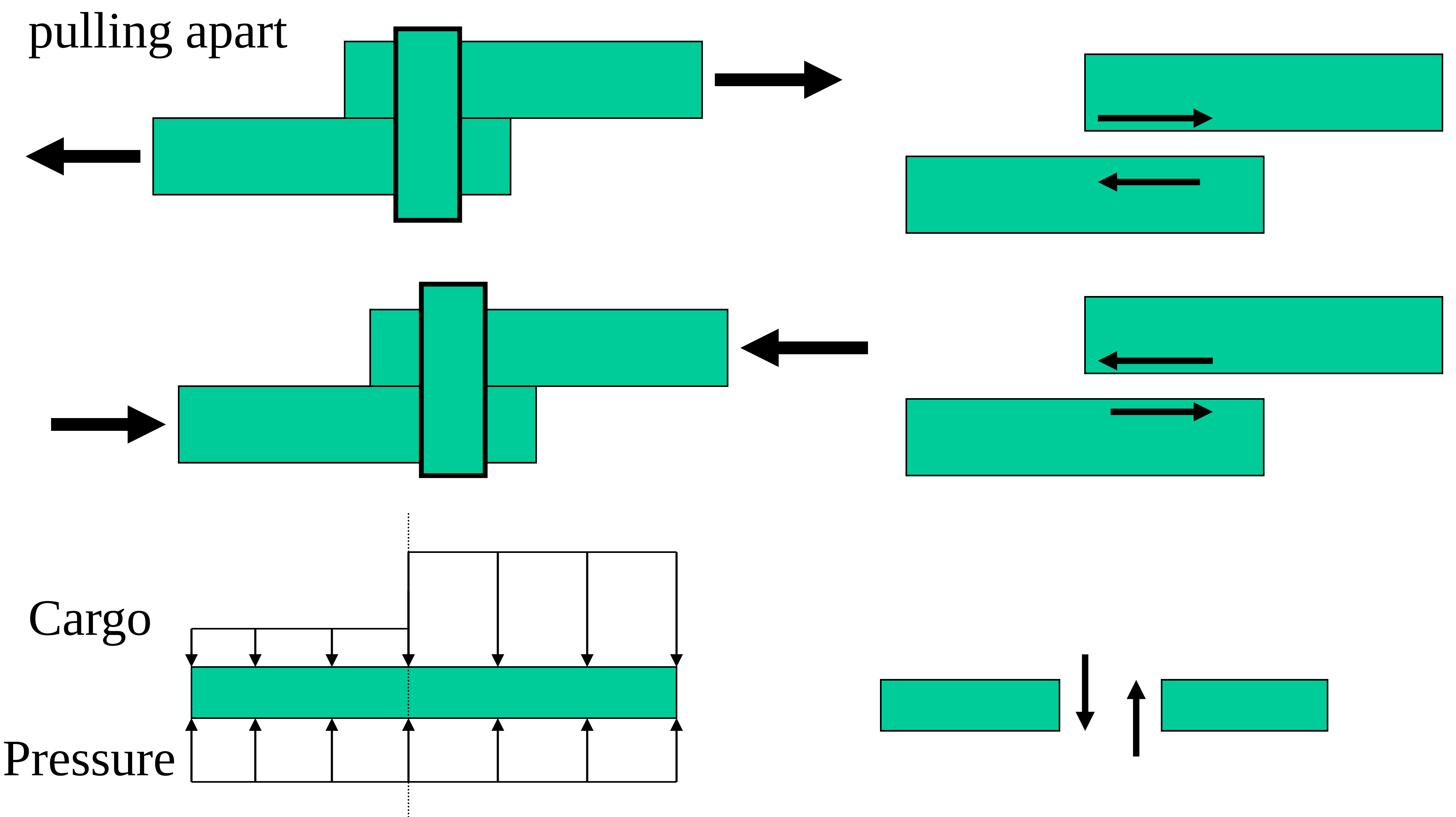
Various Loads on Materials

- **Normal Load (Axial load):** Load is perpendicular to the supporting material.
  - Tension Load: As the ends of material are pulled apart to make the material longer, the load is called a tension load.
  - Compression Load: As the ends of material are pushed in to make the material smaller, the load is called a compression load.



Loads on Materials

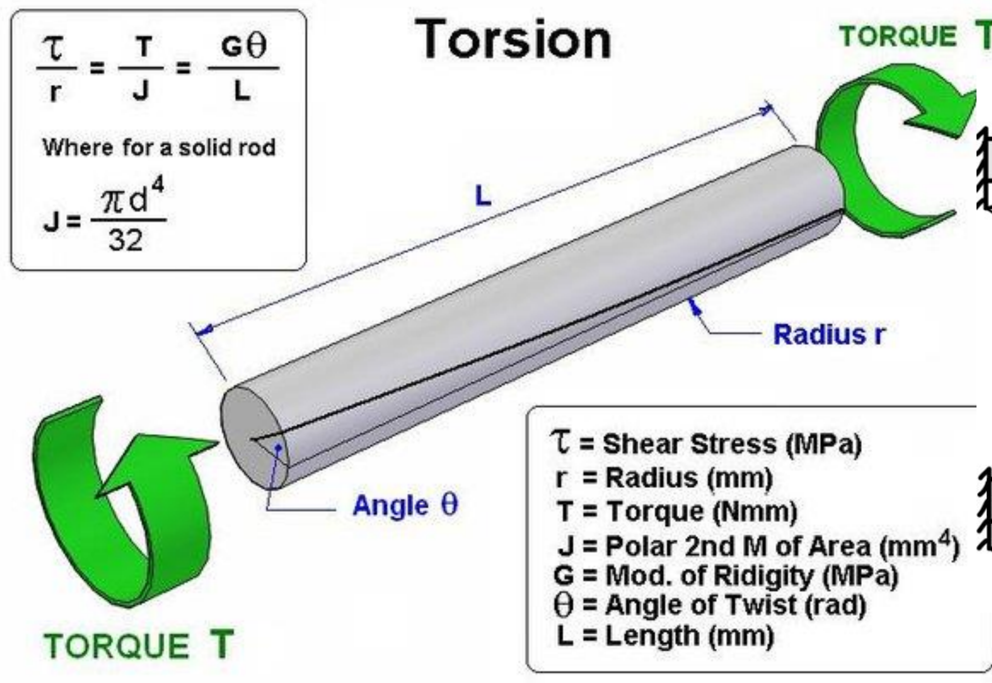
- **Shear Load : Tangential load**

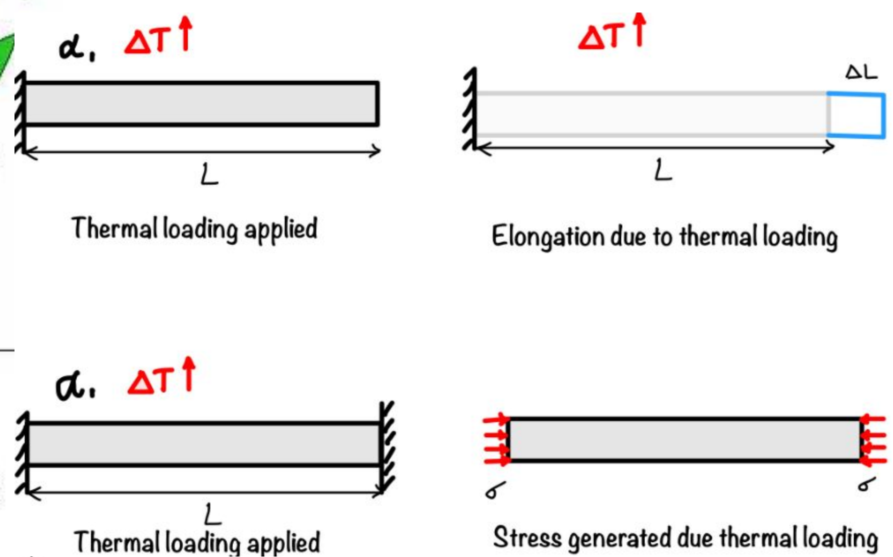


Loads on Materials

•Torsion Loads: Angular distortion on a component, such as a shaft, when a moment is applied. (Twisting)

•Thermal Loads: Distortion caused be heating or cooling a material. A normal load is created when the material is constrained in any direction in the plane that is constrained.





Stress and Strain

In order to compare materials, we must have measures.

- Stress : Internal resistance per unit Area

$$\sigma = \frac{F}{A}$$

F : Internal resistance in Newton (N)

A : cross sectional area in m^2

σ : stress in Pa (N/m^2)

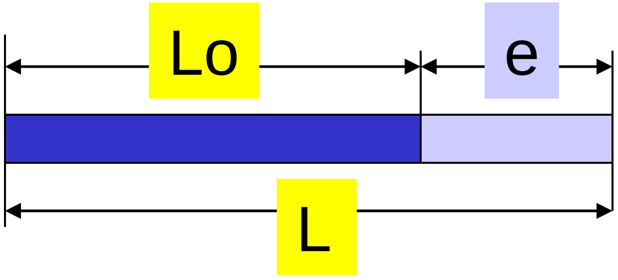
Stress and Strain

- **Strain:**

  - Ratio of change in dimensions of a material to the original dimensions

  - unit deformation

$$\varepsilon = \frac{e}{L_o}$$



e : elongation (ft)

Lo : unloaded(original) length of a material (ft)

ε : strain (ft/ft) or (in/in)

Elongation:

$$e = L - L_o$$

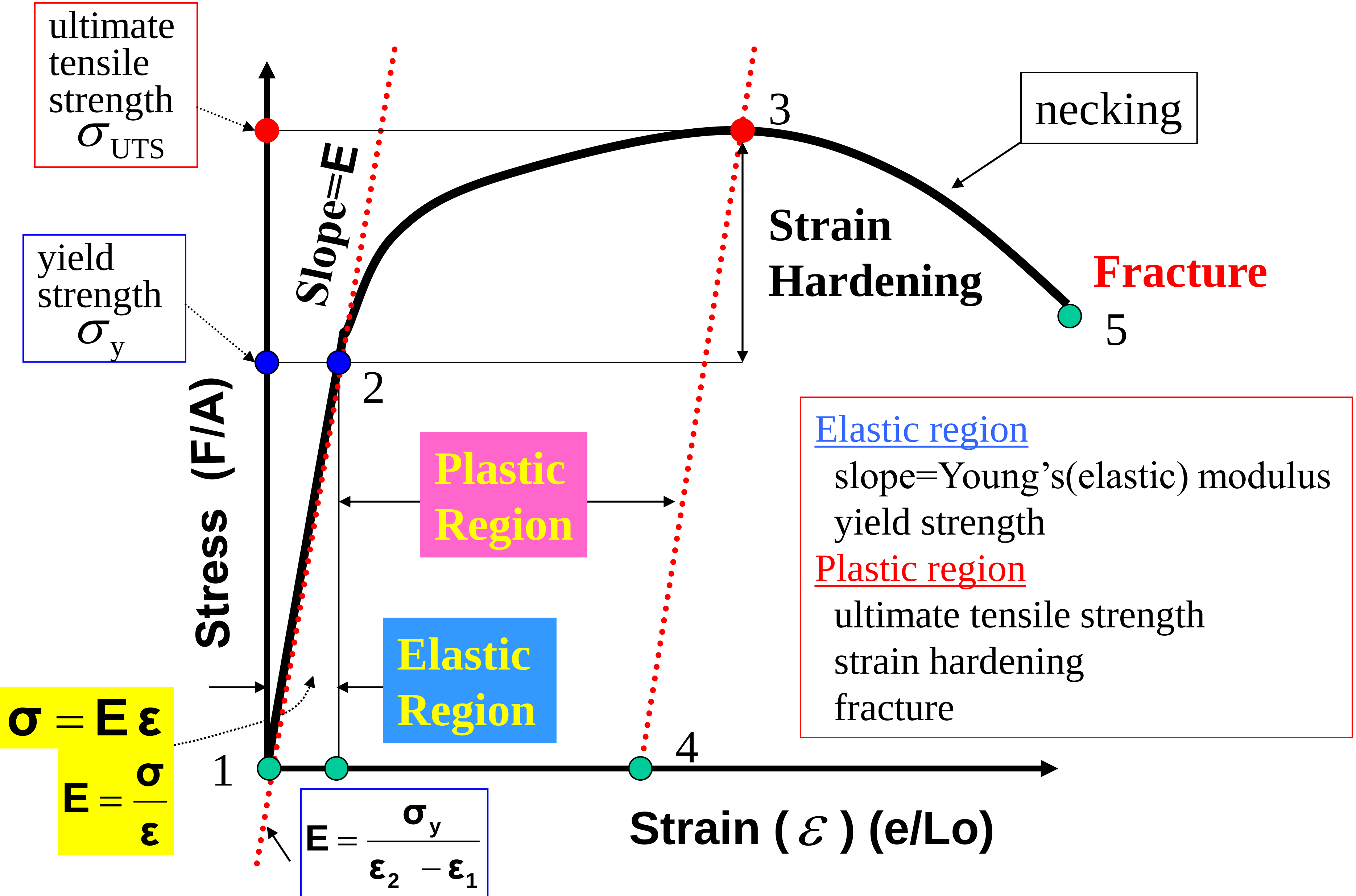
L : loaded length of a material (ft)

Baldwin Hydraulic Machine for Tension & Compression test

Stress-Strain Diagram

- A plot of Strain vs. Stress.
- The diagram gives us the behavior of the material and material properties.
- Each material produces a different stress-strain Curve.

Stress-Strain Diagram



Stress-Strain Diagram

- Elastic Region (Point 1 –2)

  - The material will return to its original shape after the material is unloaded(like a rubber band).

  - The stress is linearly proportional to the strain in this region.

$$\sigma = E\varepsilon \quad \text{or} \quad E = \frac{\sigma}{\varepsilon}$$

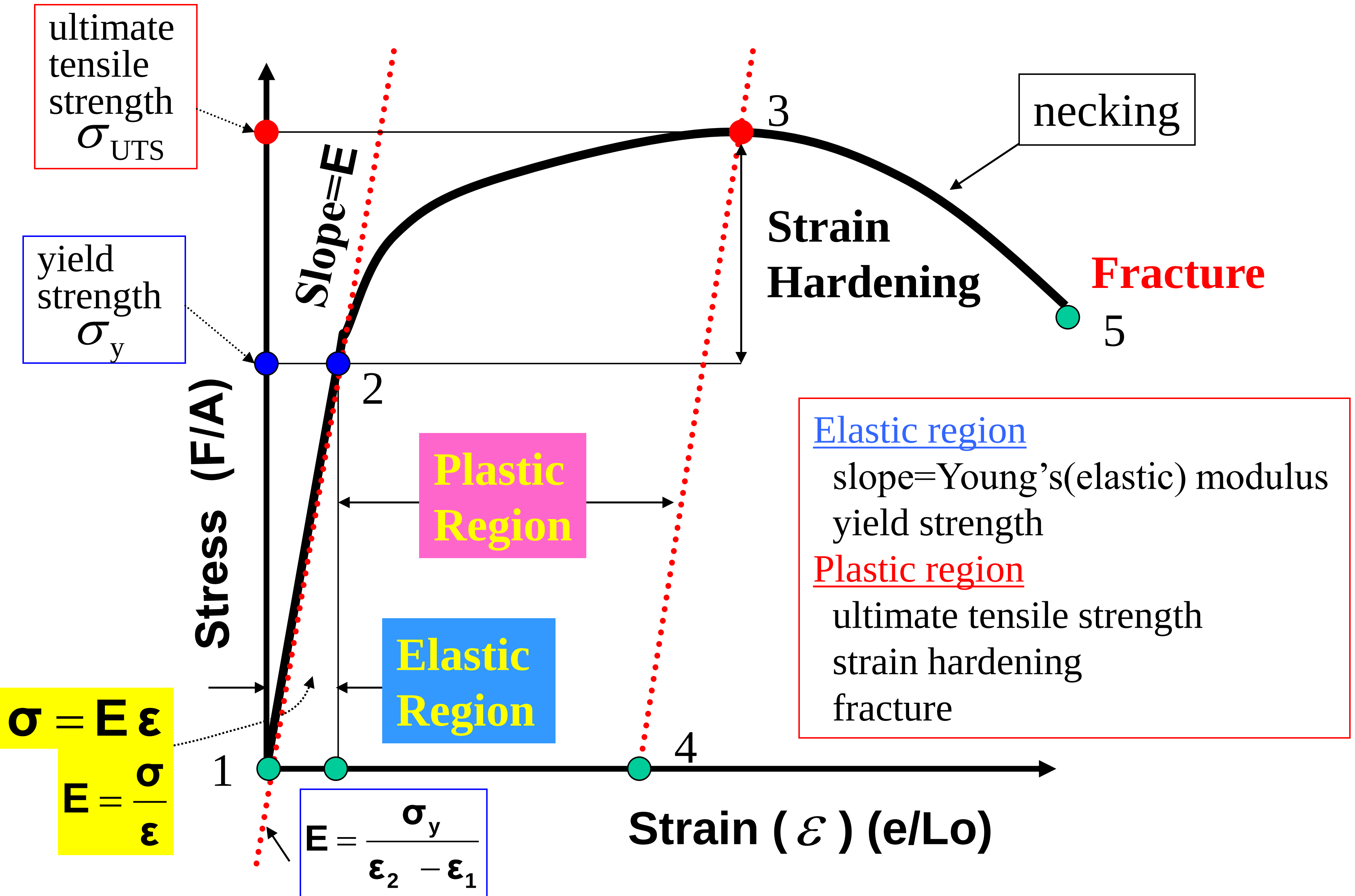
σ : Stress (psi)

E : Elastic modulus (Young's Modulus) (psi)

ε : Strain (in/in)

- Point 2 <u>Yield Strength</u> : a point at which permanent deformation occurs. (If it is passed, the material will no longer return to its original length.)

Stress-Strain Diagram



Stress-Strain Diagram

The ELASTIC Range Means:

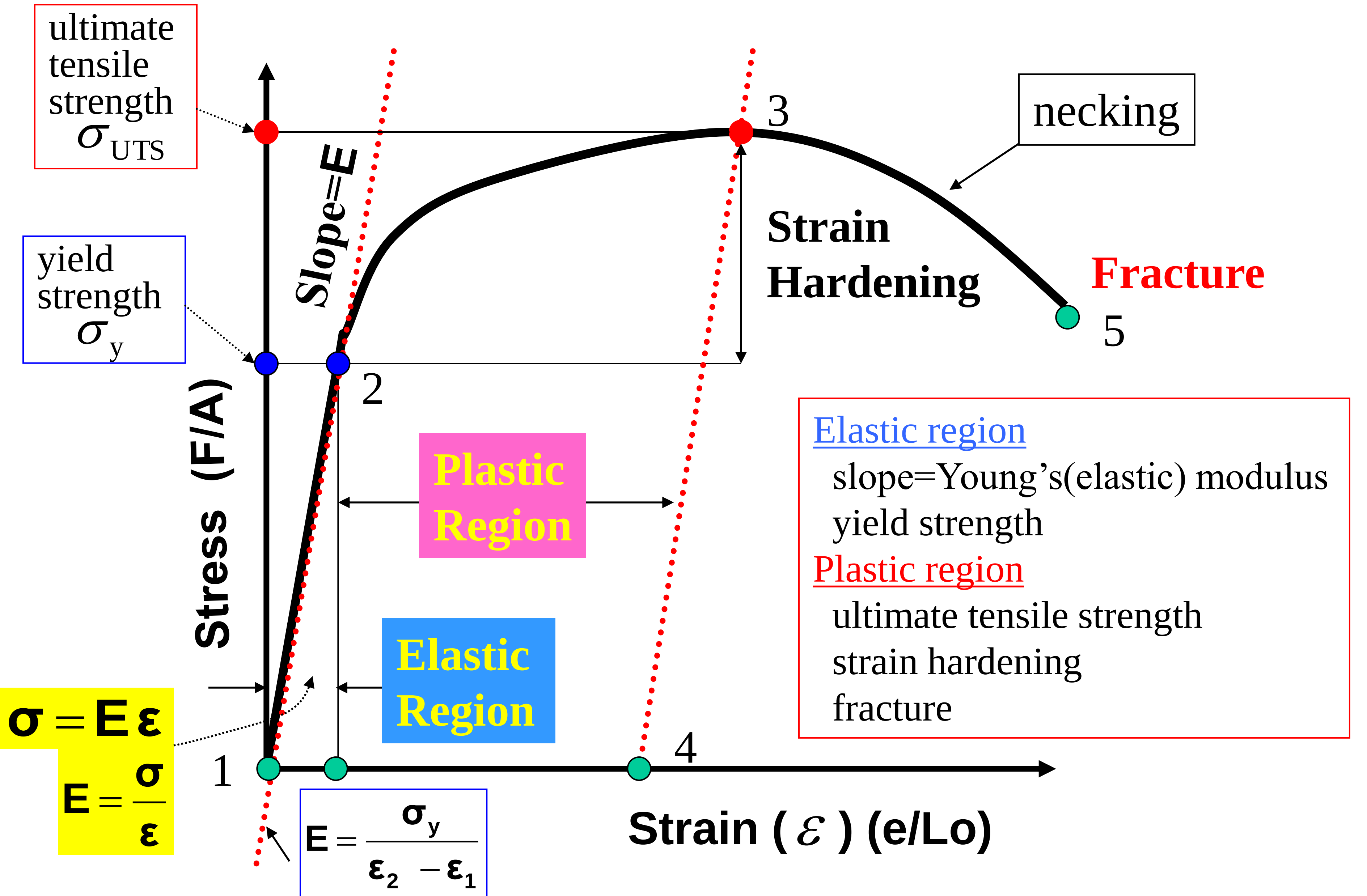
- The strain, or elongation over a unit length, will behave linearly (as in $y = mx + C$) and thus predictable.
- The material will return to its original shape (Point 1) once an applied load is removed.
- The stress within the material is less than what is required to create a plastic behavior (deform or stretch significantly without increasing stress).

Stress-Strain Diagram

Plastic Region (Point 2 –3)

- If the material is loaded beyond the yield strength, the material will not return to its original shape after unloading.
- It will have some permanent deformation.
- If the material is unloaded at Point 3, the curve will proceed from Point 3 to Point 4. The slope will be the as the slope between Point 1 and 2.
- The distance between Point 1 and 4 indicates the amount of permanent deformation.

Stress-Strain Diagram

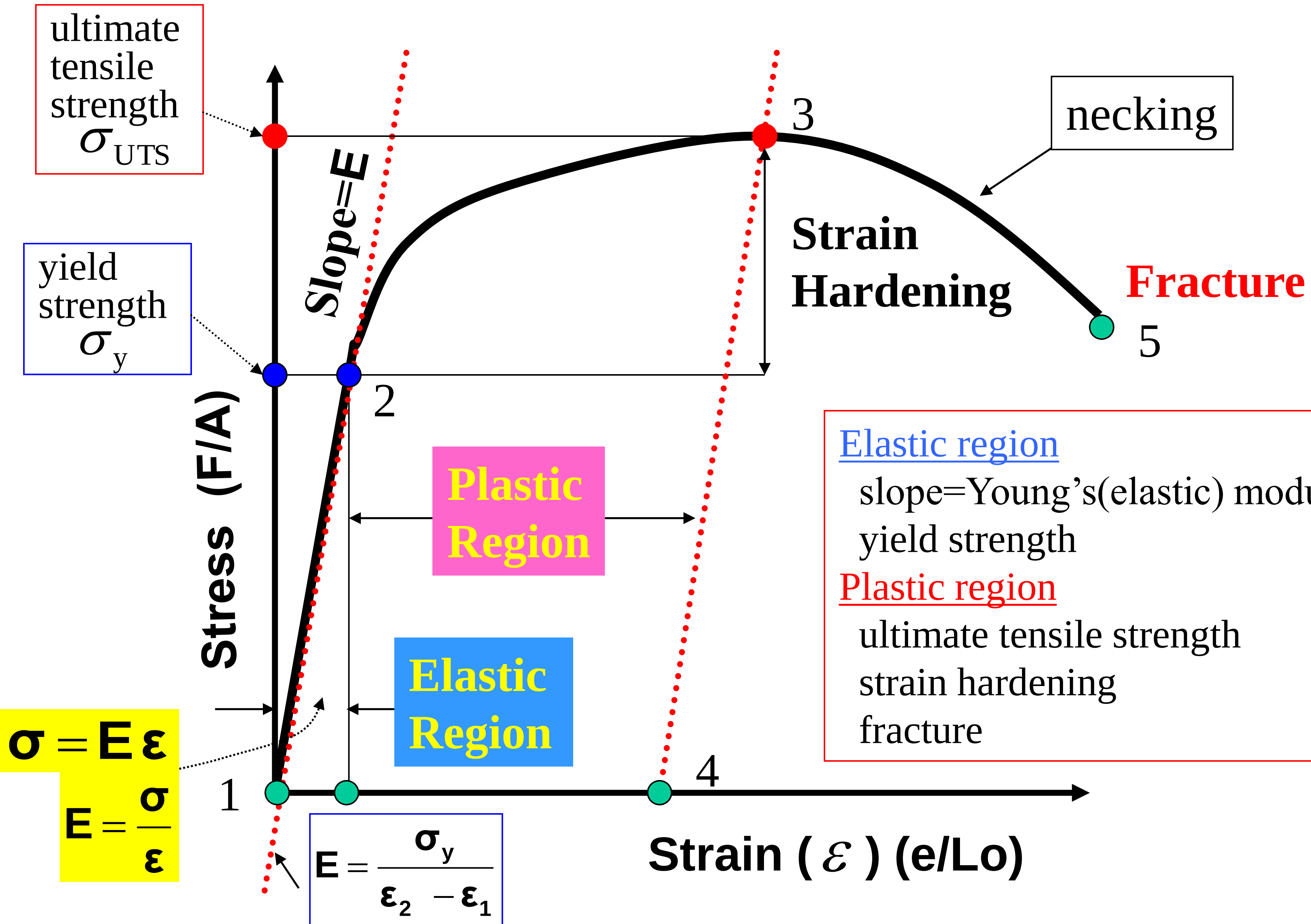


Stress-Strain Diagram

Strain Hardening

- If the material is loaded again from Point 4, the curve will follow back to Point 3 with the same Elastic Modulus(slope).
- The material now has a higher yield strength of Point 4.
- Raising the yield strength by permanently straining the material is called Strain Hardening.

Stress-Strain Diagram



Elastic region

slope=Young's(elastic) modulus

yield strength

Plastic region

ultimate tensile strength

strain hardening

fracture

Stress-Strain Diagram

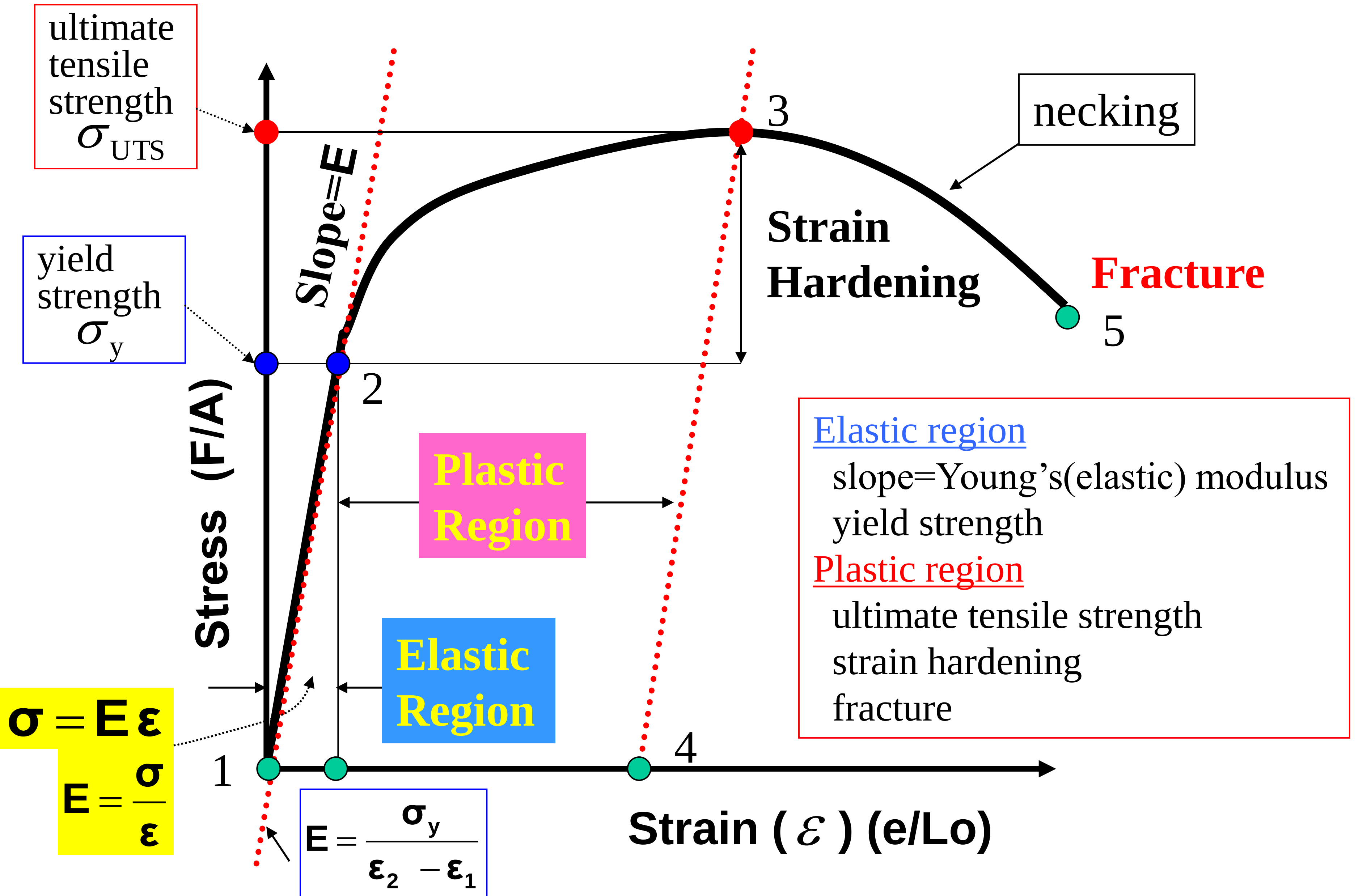
Tensile Strength (Point 3)

- The largest value of stress on the diagram is called Tensile Strength(TS) or Ultimate Tensile Strength (UTS)
- It is the maximum stress which the material can support without breaking.

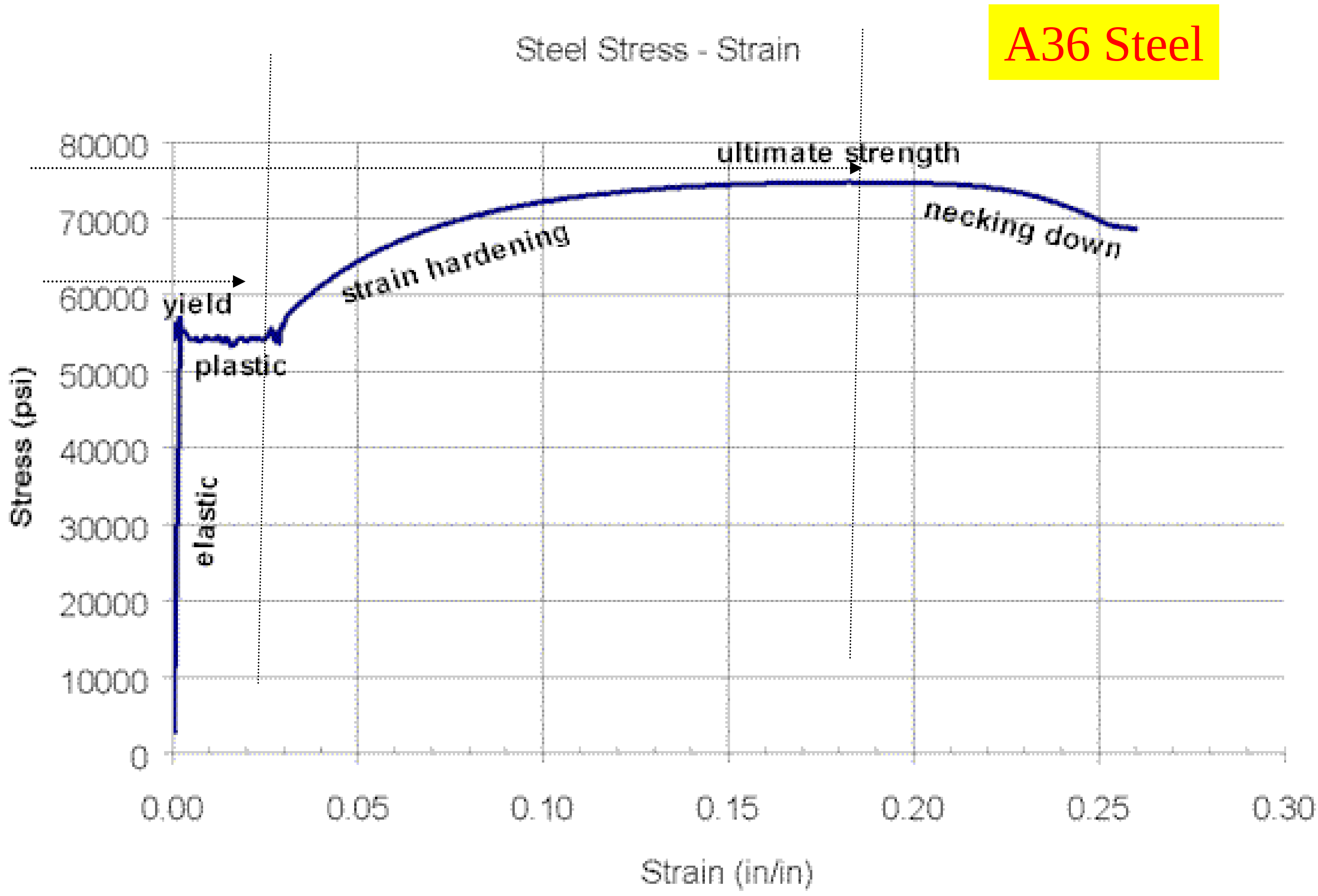
Fracture (Point 5)

- If the material is stretched beyond Point 3, the stress decreases as necking and non-uniform deformation occur.
- Fracture will finally occur at Point 5.

Stress-Strain Diagram



Stress-Strain Diagram



Material Properties

Characteristics of Material are described as

- **Strength**
- **Hardness**
- **Ductility**
- **Brittleness**
- **Toughness**

Material Properties

Strength:

- Measure of the material property to resist deformation and to maintain its shape
- It is quantified in terms of yield stress σ_y or ultimate tensile strength σ_{ult}.
- High carbon steels and metal alloys have higher strength than pure metals.
- Ceramic also exhibit high strength characteristics.

Material Properties

Hardness:

- Measure of the material property to resist indentation, abrasion and wear.

- It is quantified by hardness scale such as Rockwell and Brinell hardness scale that measure indentation / penetration under a load.

- Hardness and Strength correlate well because both properties are related to inter-molecular bonding. A high-strength material is typically resistant to wear and abrasion.

A comparison of hardness of some typical materials:

Material	Brinell Hardness
Pure Aluminum	15
Pure Copper	35
Mild Steel	120
304 Stainless Steel	250
Hardened Tool Steel	650/700
Hard Chromium Plate	1000
Chromium Carbide	1200
Tungsten Carbide	1400
Titanium Carbide	2400
Diamond	8000
Sand	1000

Material Properties

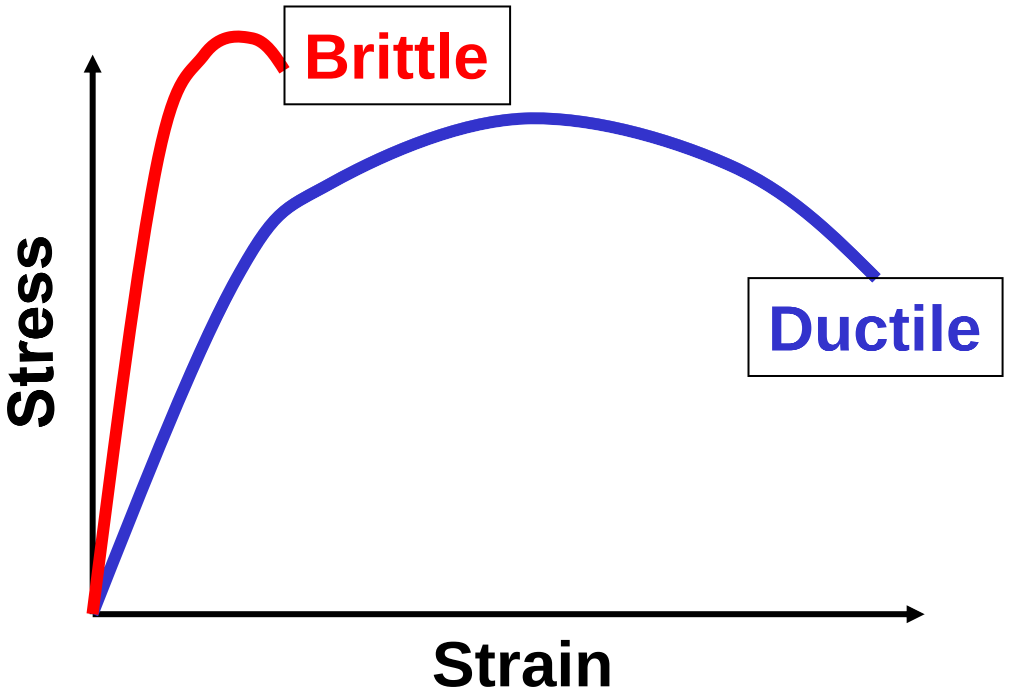
Ductility:

- Measure of the material property to deform before failure.
- It is quantified by reading the value of strain at the fracture point on the stress strain curve.
- Ductile materials can be pulled or drawn into pipes, wire, and other structural shapes
- Examples of ductile material :
  - low carbon steel
  - aluminum
  - copper
  - brass

Material Properties

Brittleness:

- Measure of the material's inability to deform before failure.
- The opposite of ductility.
- Example of Brittle material : glass, high carbon steel, ceramics



Material Properties

Toughness:

- Measure of the material ability to absorb energy.
- It is measured by two methods.

a) Integration of stress strain curve

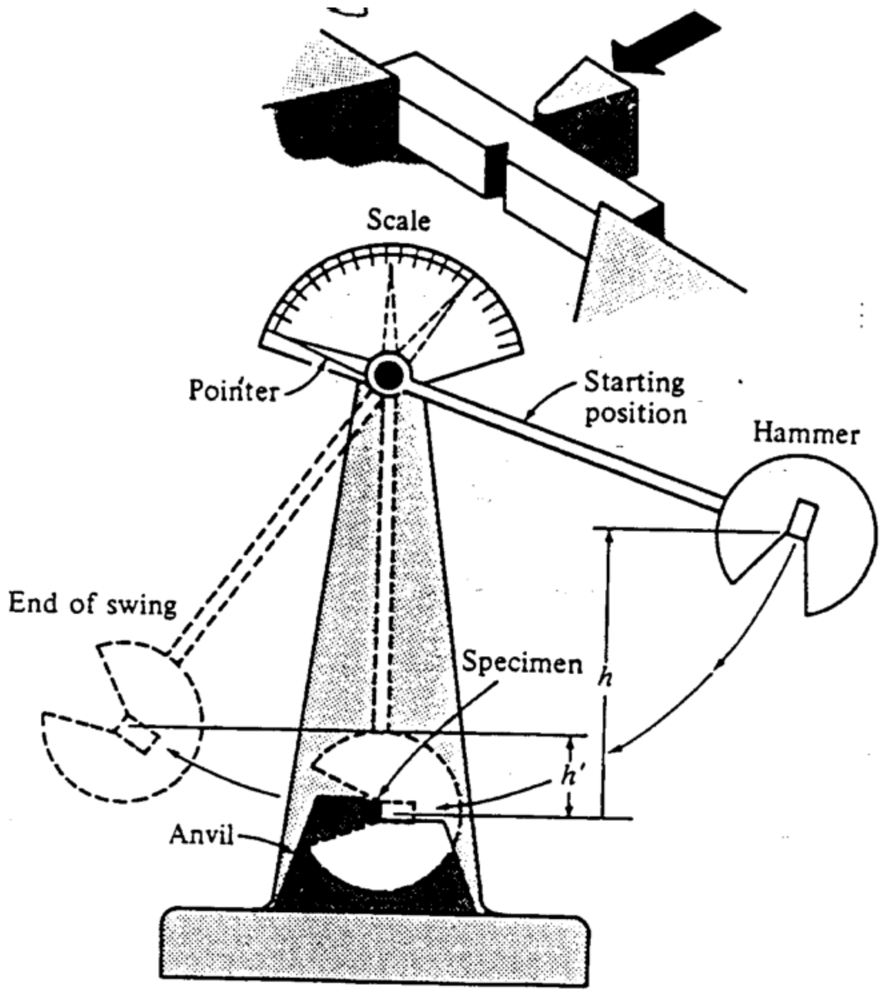
- Slow absorption of energy
- Absorbed energy per unit volume

b) Charpy test

- Ability to absorb energy of an impact without fracturing.
- Impact toughness can be measured.

Material Properties

Charpy V-Notch Test:



Material Properties

Charpy V-Notch Test:

- Charpy test is an impact toughness measurement test because the energy is absorbed by the specimen very rapidly.

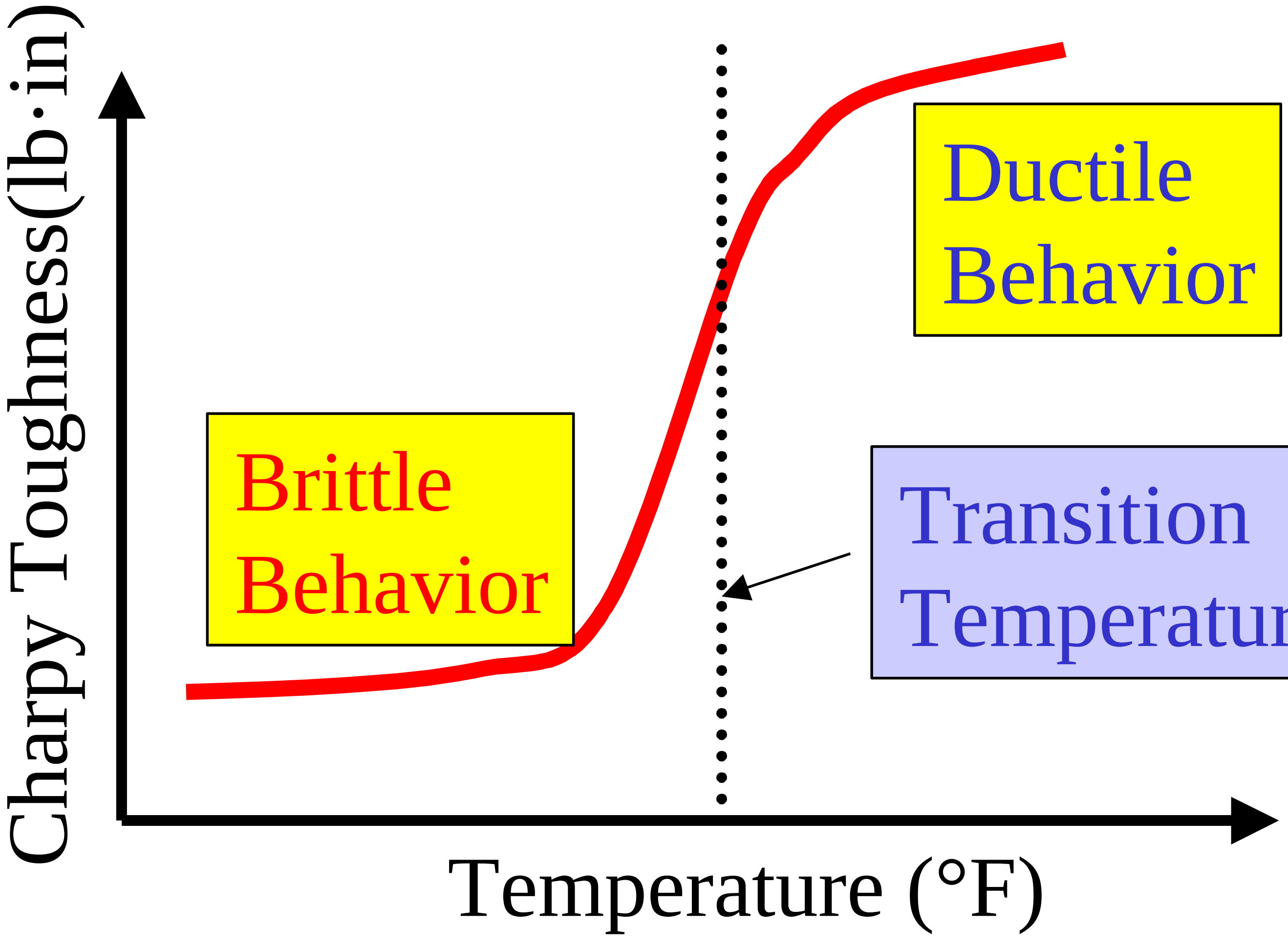
- The potential energy of the pendulum before and after impact can be calculated form the initial and final location of the pendulum.

- The potential energy difference is the energy it took to break the material absorbed during the impact.

- Purpose is to evaluate the impact toughness as a function of temperature

Material Properties

Charpy V-Notch Test:

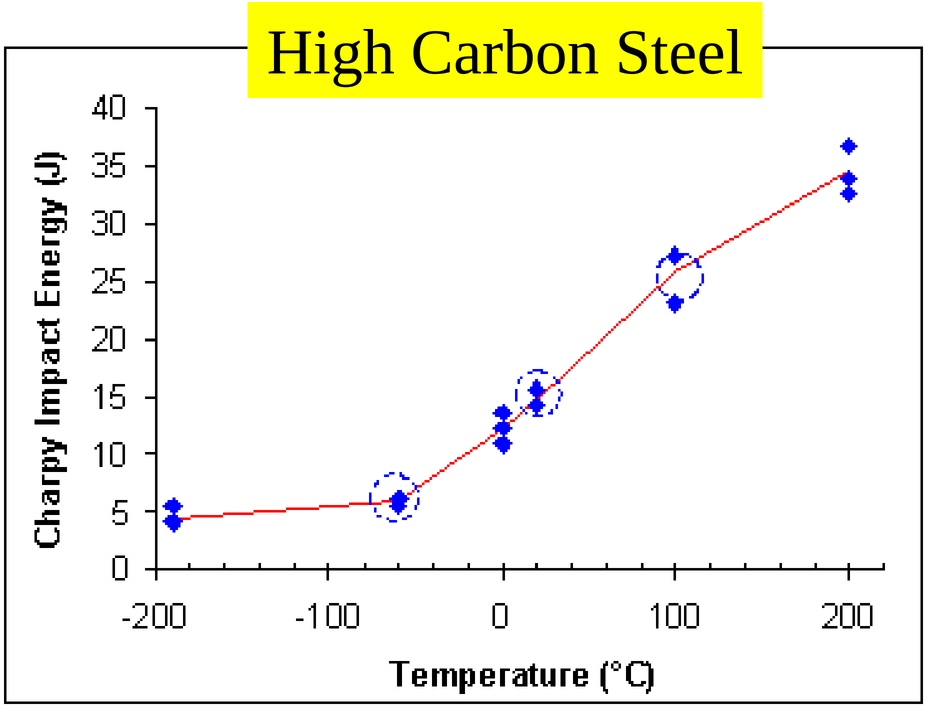


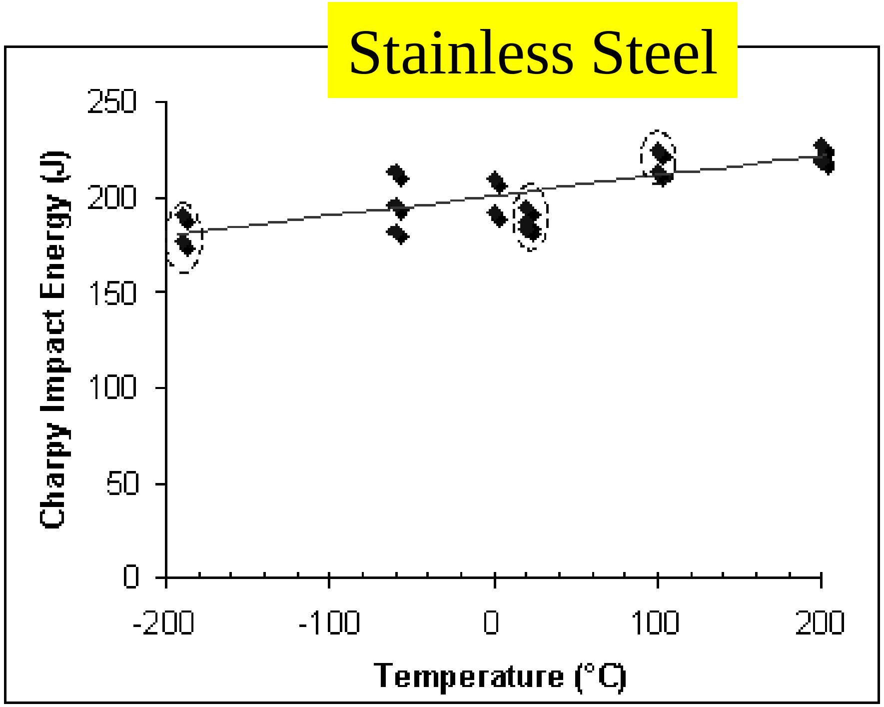
Material Properties

Charpy V-Notch Test:

- At low temperature, where the material is brittle and not strong, little energy is required to fracture the material.
- At high temperature, where the material is more ductile and stronger, greater energy is required to fracture the material
- The transition temperature is the boundary between brittle and ductile behavior.
- The transition temperature is an extremely important parameter in selection of construction material.

Charpy V-Notch Test:



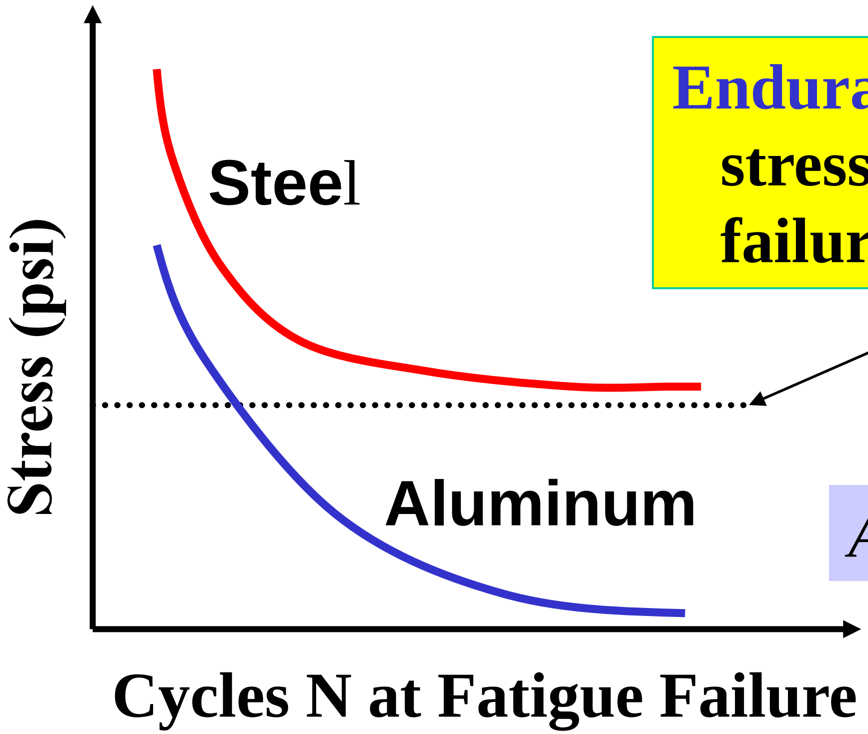


Material Properties

Fatigue:

- The repeated application of stress typically produced by an oscillating load such as vibration.
- Sources of ship vibration are engine, propeller and waves.

MAXIMUM stress decreases as the number of loading cycles increases.



Endurance Limit : A certain threshold stress which will not cause the fatigue failure for the number of cycles.

Aluminum has no endurance limit

Factors effecting Material Properties

Temperature :

Increasing temperature will:

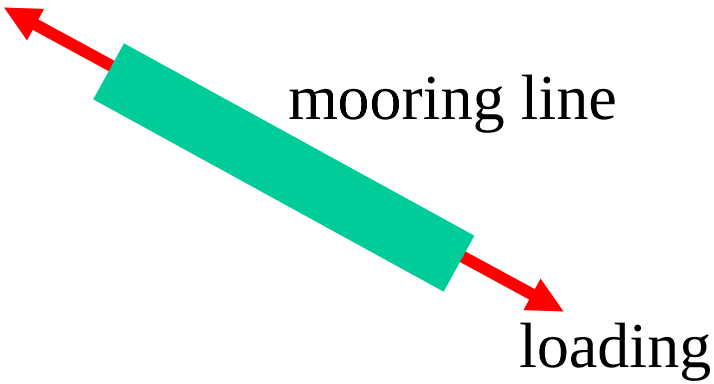
- Decrease Modulus of Elasticity
  (As Long as Structure Does Not Change)
- Decrease Yield Strength
- Decrease Ultimate Tensile Strength
- Decrease Hardness
- Increase Ductility
- Decrease Brittleness

Environment:

- Sulfites, Chlorine, Oxygen in water,
  Radiation, Pressure

Example:

Mooring line length =100 ft
diameter=1.0 in
Axial loading applied=25,000 lb
Elongation due to loading=1.0 in



1) Find the normal stress.

$$\sigma = \frac{F}{A} = \frac{25{,}000\ \ lb}{0.785\ \ in^2} = 31{,}800\ \ psi$$

$$\mathrm{A} = \pi\ \ \mathrm{r}^2 = \pi\,(0.5in)^2 = 0.785\ \ in^2$$

2) Find the strain.

$$\varepsilon = \frac{e}{L_o} = \frac{1in}{100\,ft \times \frac{12in}{1\,ft}} = 0.00083\ \ (in\,/\,in)$$

Example:

- Salvage crane is lifting an object of 20,000 lb.
- Characteristics of the cable
  diameter=1.0 in, length prior to lifting =50 ft

$\sigma_y = 60{,}000$ psi

$\sigma_{UT} = 70{,}000$ psi

$E = 35 \times 10^6$ psi

1) Find the normal stress in the cable.

$$\sigma = \frac{F}{A} = \frac{20{,}000\ lb}{0.785\ in^2} = 25{,}478\ psi$$

$$(\mathrm{A} = \pi\ \mathrm{r}^2 = \pi (0.5\ in)^2 = 0.785\ in^2)$$

2) Find the strain.

$$\varepsilon = \frac{\sigma}{E} = \frac{25{,}478\ psi}{35 \times 10^6\ psi} = 0.000728\ (in/in)$$

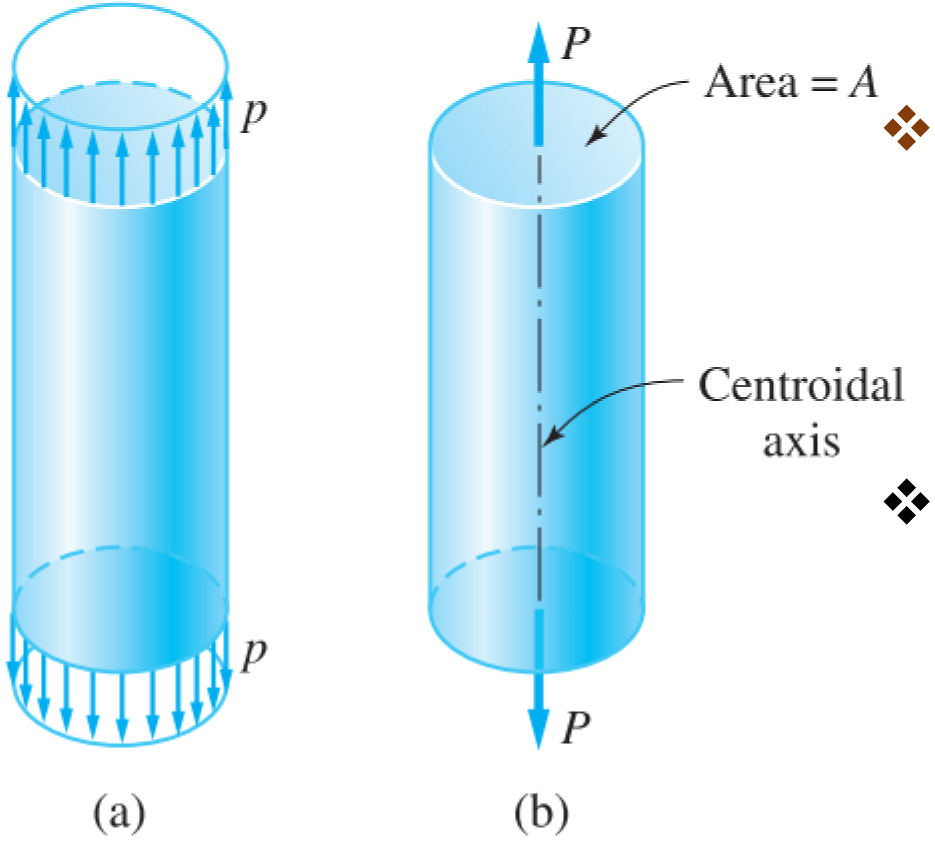
3) Determine the cable stretch in inches.

$$\varepsilon = \frac{e}{L_o}$$

$$e = \varepsilon \times L_o = (0.000728\ in/in) \times (50\ ft \times \frac{12 in}{1 ft}) = 0.44\ in$$

Stress

Axial or Normal Stress

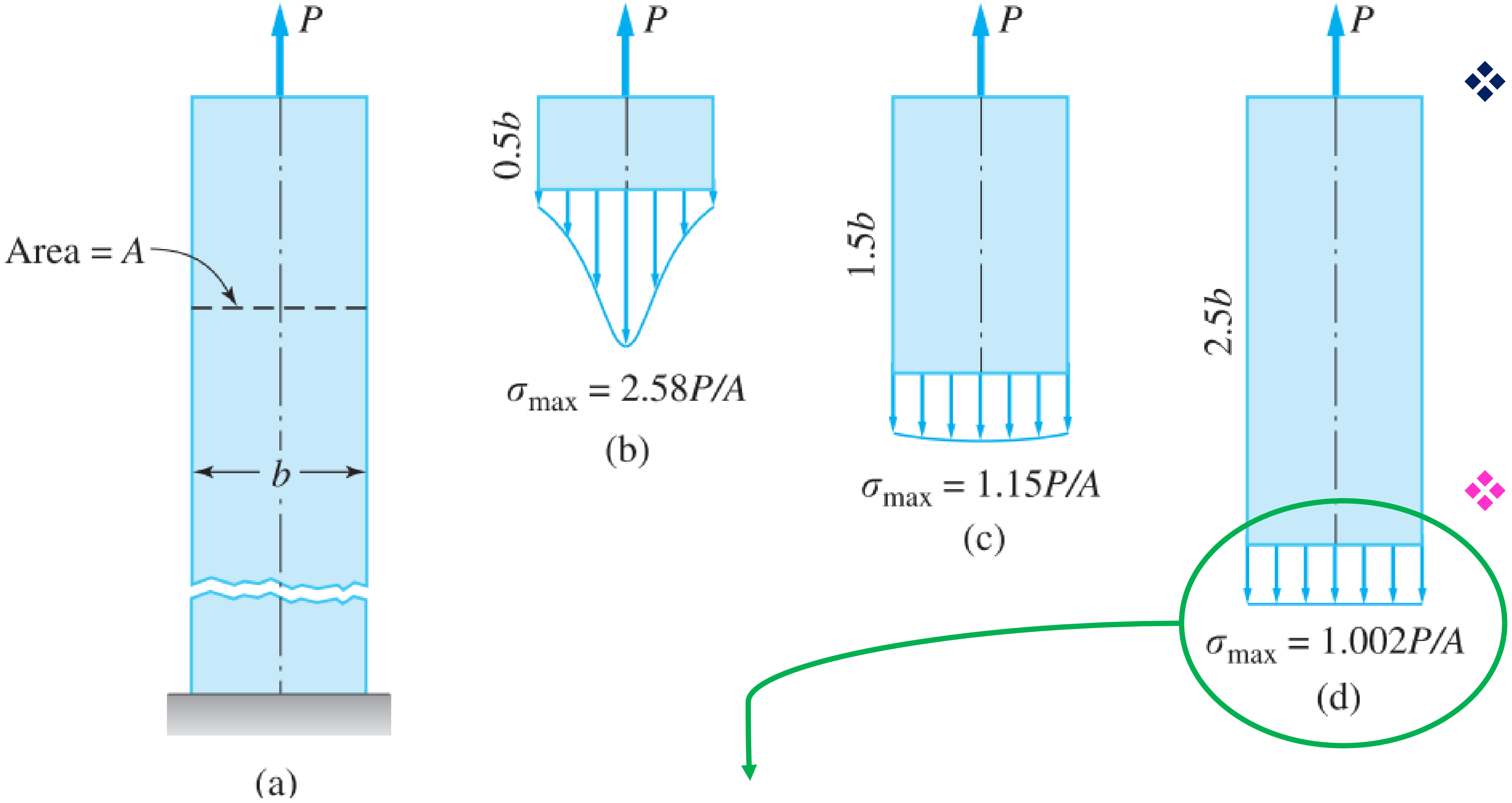


- When the loading is uniform, its resultant passes through the centroid of the loaded area.
- The internal forces acting on all cross sections are also uniformly distributed. Therefore, the normal stress acting at any point on a cross section is

$$\sigma = \frac{P}{A}$$

- Advanced methods of analysis show that on cross sections close to the ends, the maximum stress is considerably higher than the average stress $\frac{P}{A}$. As we move away from the ends, the stress becomes more uniform

Saint Venant’s principle



- ❖ As an example of concentrated loading, consider the thin strip of width b shown in Fig. (a). The strip is loaded by the centroidal force P.
- ❖ Figures (b)–(d) show the stress distribution on three different cross sections.

Note that at a distance 2.5b from the loaded end, the maximum stress differs by only 0.2% from the average stress $\frac{P}{A}$.

Saint Venant’s principle:

“The difference between the effects of two different but statically equivalent loads becomes very small at sufficiently large distances from the load.”

The example in previous Figure is an illustration of Saint Venant’s principle.

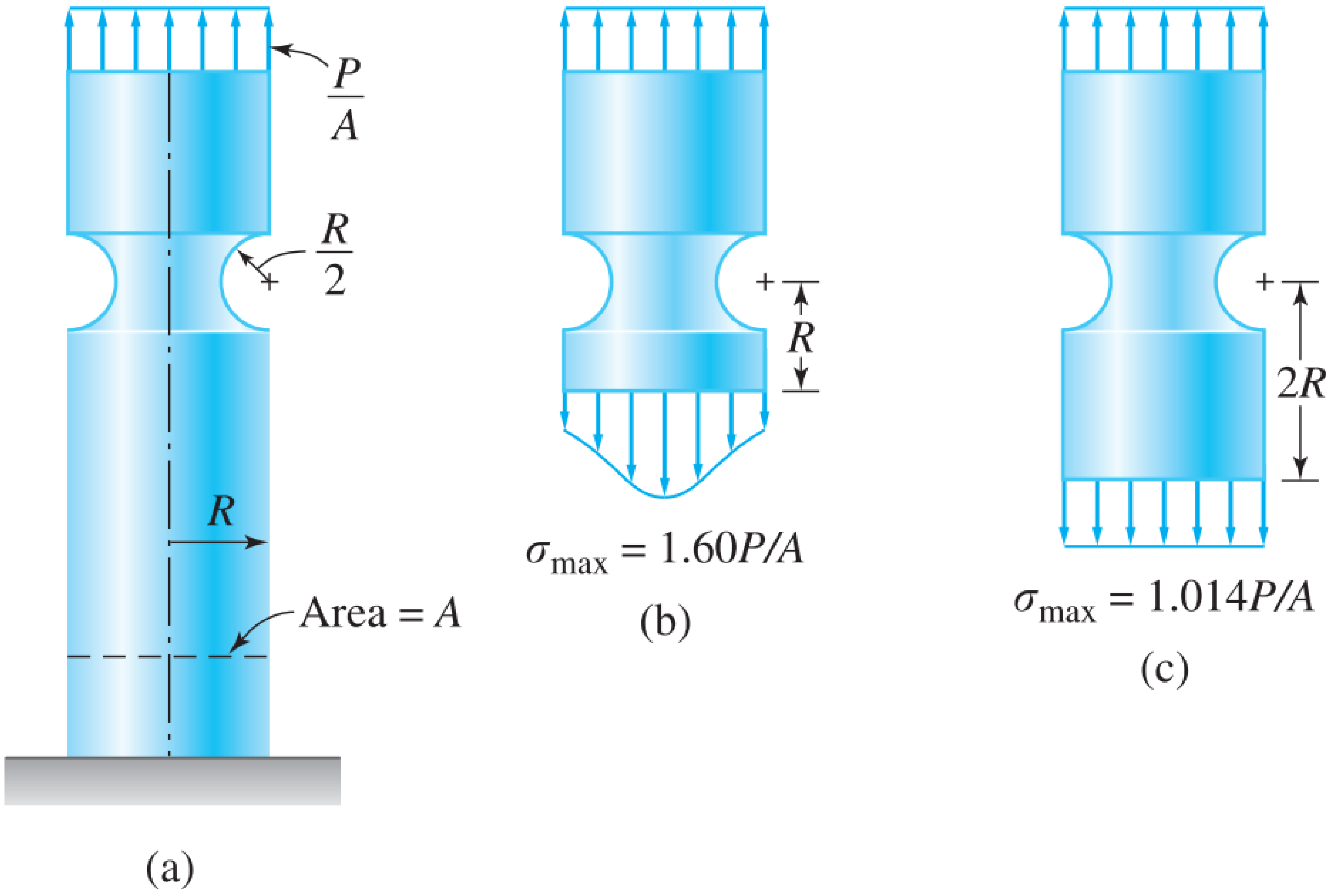
The principle also applies to the effects caused by abrupt changes in the cross section.

A grooved cylindrical bar of radius R is loaded with uniformly distributed load P over the end of the bar.

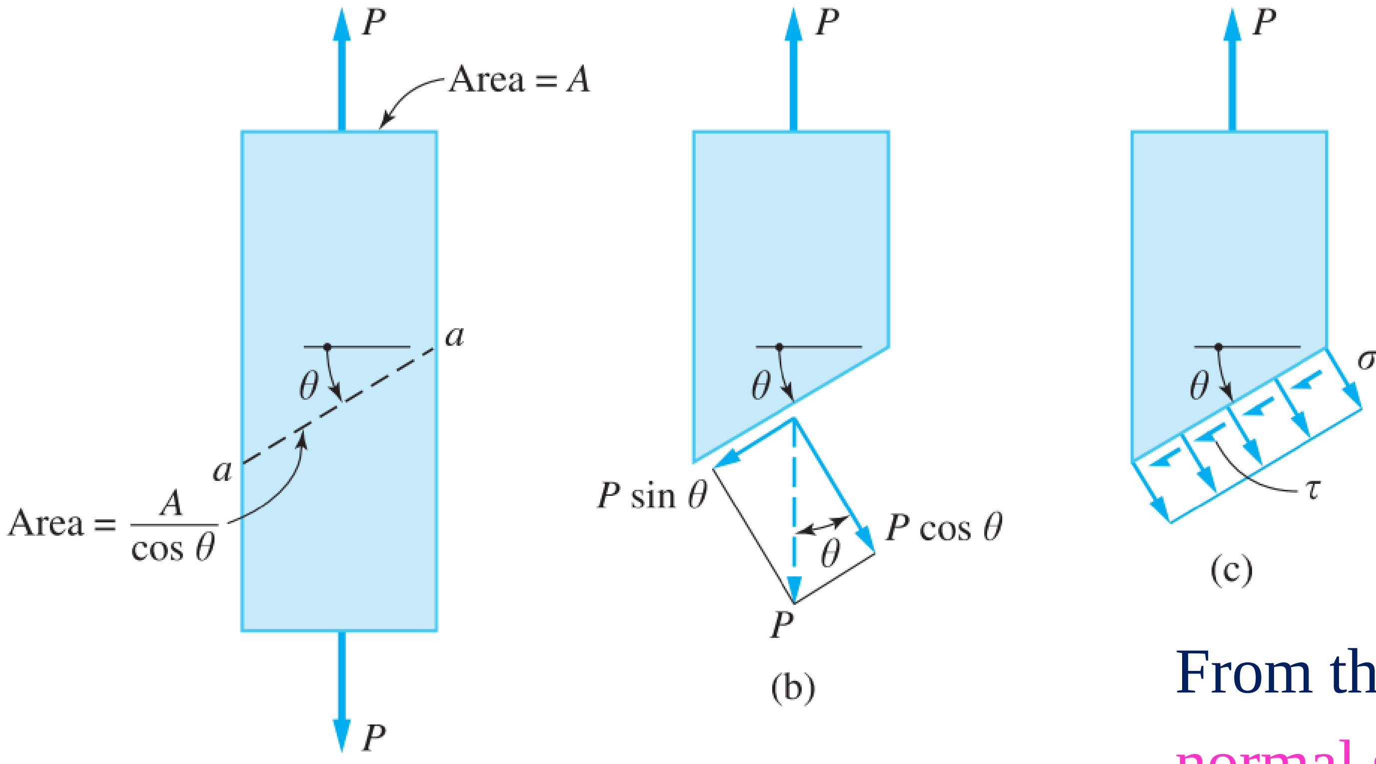
If the groove were not present, the normal stress acting at all points on a cross section would be $\frac{P}{A}$.

Introduction of the groove disturbs the uniformity of the stress, but this effect is confined to the vicinity of the groove, as seen in Figures given in next slide.

Saint Venant's principle



- ❖ We often replace loads (including support reactions) by their resultants and ignore the effects of holes, grooves, and fillets on stresses and deformations.
- ❖ Many of the simplifications are not only justified but necessary. Without simplifying assumptions, analysis would be exceedingly difficult.
- ❖ However, we must always keep in mind the approximations that were made, and make allowances for them in the final design.



When a bar of cross-sectional area A is subjected to an axial load P, the normal stress $\frac{P}{A}$ acts on the cross section of the bar.

Let us now consider the stresses that act on plane a-a that is inclined at the angle θ to the cross section, as shown in Figure.

$$\sigma = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta$$

$$\tau = \frac{P\sin\theta}{A/\cos\theta} = \frac{P}{A}\sin\theta\cos\theta = \frac{P}{2A}\sin 2\theta$$

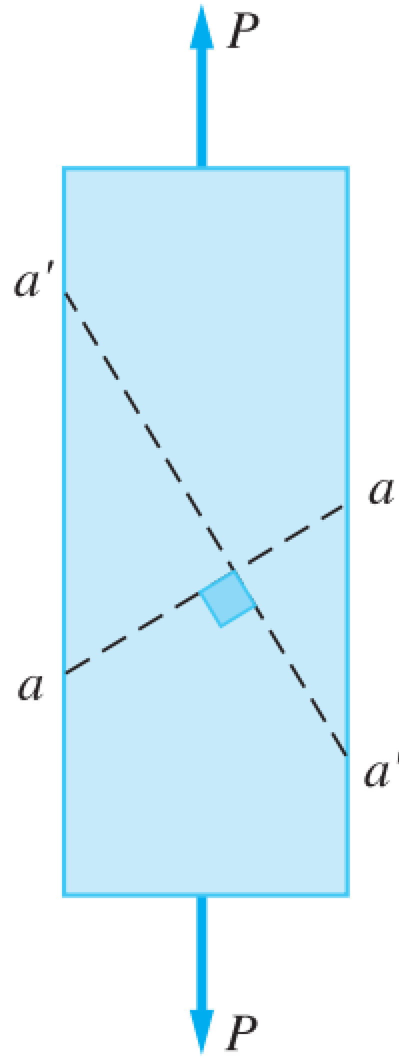
From these equations we see that the maximum normal stress is $\frac{P}{A}$, and it acts on the cross section of the bar (that is, on the plane $\theta = 0°$). The shear stress is zero when $\theta = 0°$, as would be expected.

The maximum shear stress is $\frac{P}{2A}$, which acts on the planes inclined at $\theta = 45°$ to the cross section.

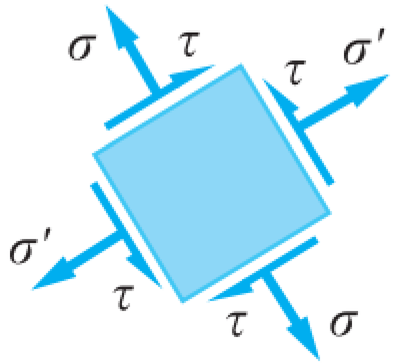
Concept of Complementary stresses

- In summary, an axial load causes not only normal stress but also shear stress. The magnitudes of both stresses depend on the orientation of the plane on which they act.
- By replacing θ with θ = 90° in equations, we obtain the stresses acting on plane a’ –a’ , which is perpendicular to a-a, as illustrated in Figure (a).
- We obtained modified equation of normal and shear stresses in plane a’ –a’ .

$$\sigma' = \frac{P}{A}\sin^2\theta \qquad \tau' = -\frac{P}{2A}\sin 2\theta$$



(a)



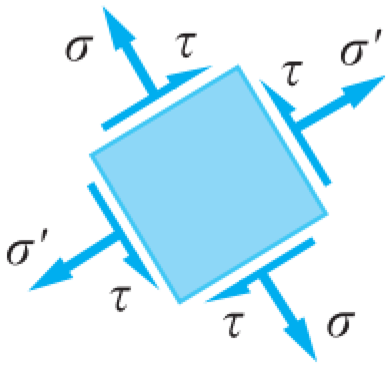
(b)

Complementary stresses

- In case of plane a’ –a’, the identities

$\cos(\theta + 90^\circ) = -sin\theta$ and $\sin2(\theta + 90^\circ) = -sin2\theta$.

- Because the stresses act on mutually perpendicular, or “complementary” planes, they are called complementary stresses.
- The traditional way to visualize complementary stresses is to draw them on a small (infinitesimal) element of the material, the sides of which are parallel to the complementary planes, as in Figure.



$$\tau' = -\tau$$

- In other words, The shear stresses that act on complementary planes have the same magnitude but opposite sense.

Factor of Safety

- The design of axially loaded bars is usually based on the maximum normal stress in the bar. Thus, the maximum normal stress

$$\sigma = \frac{P}{A}$$

σ must not exceed the working stress of the material from which the bar is to be fabricated.

- The working stress, also called the allowable stress, is the largest value of stress that can be safely carried by the material. Working stress, denoted by σ_w.

$$Factor\ of\ safety = \frac{\sigma}{\sigma_w}$$

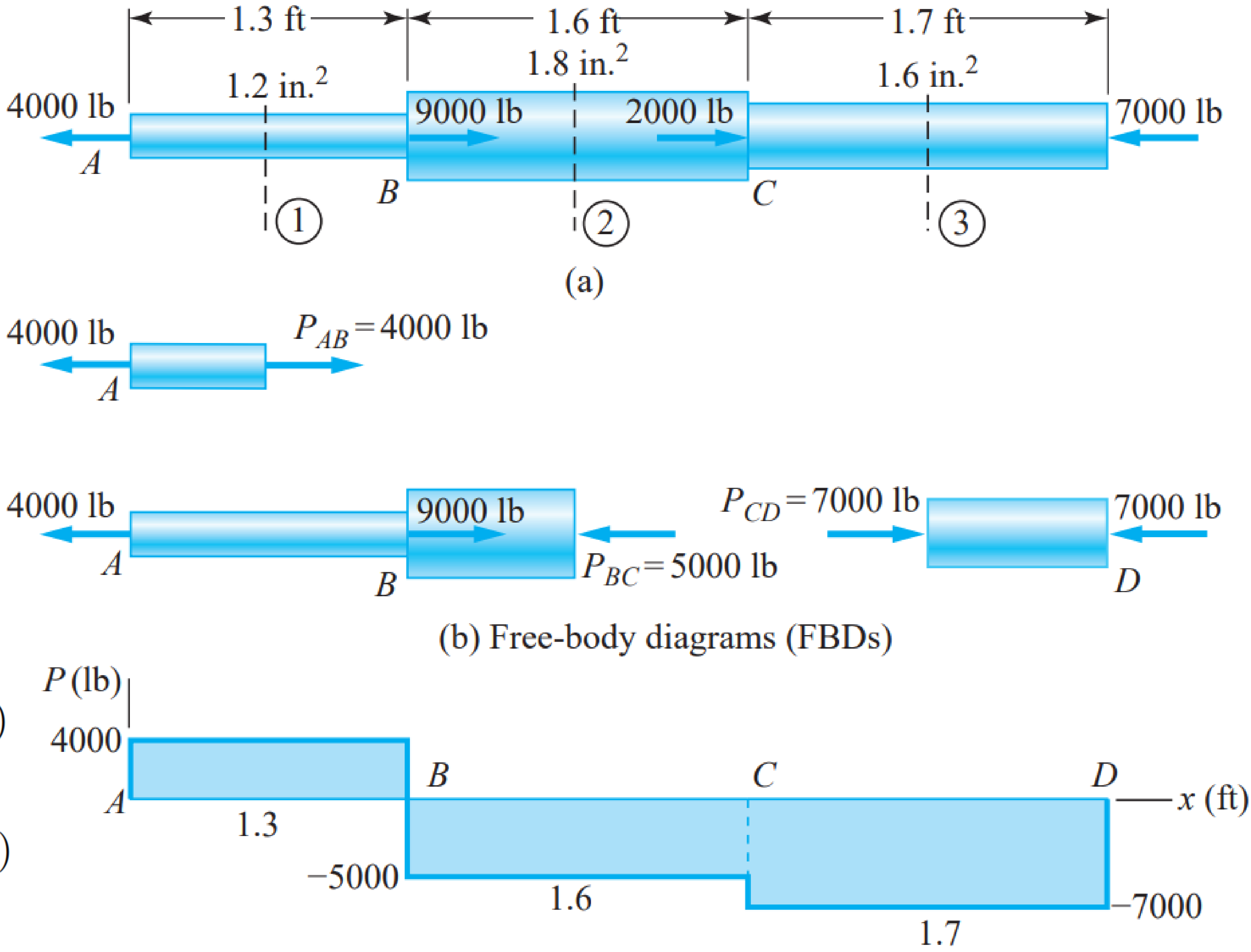
$$\sigma = Yield\ stress\ for\ ductile\ materials$$

$$\sigma = Ultimate\ stress\ for\ ductile\ materials$$

Selection of Factor of Safety

- Many factors must be considered when selecting the working stress. This selection should not be made by the novice; usually the working stress is set by a group of experienced engineers and is embodied in building codes and specifications.
- A discussion of the factors governing the selection of a working stress starts with the observation that in many materials the proportional limit is about one-half the ultimate strength. To avoid accidental overloading, a working stress of one-half the proportional limit is usually specified for dead loads that are gradually applied. (The term dead load refers to the weight of the structure and other loads that, once applied, are not removed.)
- A working stress set in this way corresponds to a factor of safety of 4 with respect to σ_{ult} and is recommended for materials that are known to be uniform and homogeneous.
- For other materials, such as wood, in which unpredictable nonuniformities (such as knotholes) may occur, larger factors of safety are used. The dynamic effect of suddenly applied loads also requires higher factors of safety.

Que. 1. The bar ABCD in Fig. (a) consists of three cylindrical steel segments with different lengths and cross-sectional areas. Axial loads are applied as shown. Calculate the normal stress in each segment.



(a)

(b) Free-body diagrams (FBDs)

(c) Axial force diagram (tension assumed positive)

$$\sigma_{AB} = \frac{P_{AB}}{A_{AB}} = \frac{4000 \text{ lb}}{1.2 \text{ in.}^2} = 3330 \text{ psi (T)}$$

$$\sigma_{BC} = \frac{P_{BC}}{A_{BC}} = \frac{5000 \text{ lb}}{1.8 \text{ in.}^2} = 2780 \text{ psi (C)}$$

$$\sigma_{CD} = \frac{P_{CD}}{A_{CD}} = \frac{7000 \text{ lb}}{1.6 \text{ in.}^2} = 4380 \text{ psi (C)}$$

Que.2. Figure (a) shows a two-member truss supporting a block of weight W. The cross-sectional areas of the members are 800 mm^2 for AB and 400 mm^2 for AC. Determine the maximum safe value of W if the working stresses are 110 MPa for AB and 120 MPa for AC

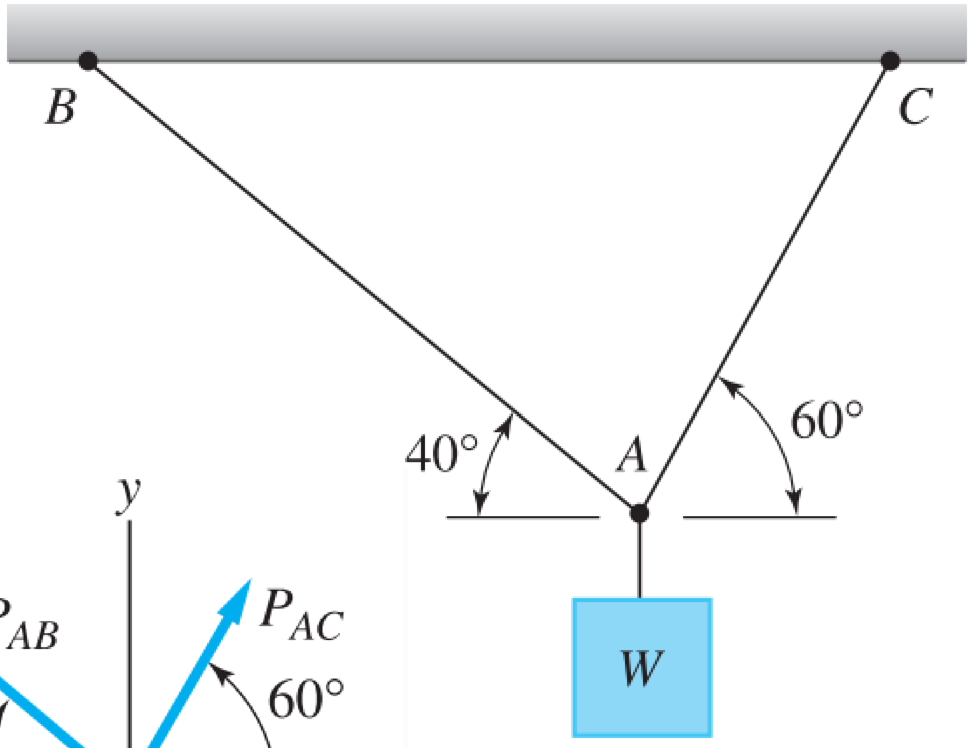
Sol: From free body diagram as in Fig. (b) equilibrium equation can be written as

$$\sum F_x = 0 \quad \overset{+}{\rightarrow} \quad P_{AC}\cos 60^\circ - P_{AB}\cos 40^\circ = 0$$

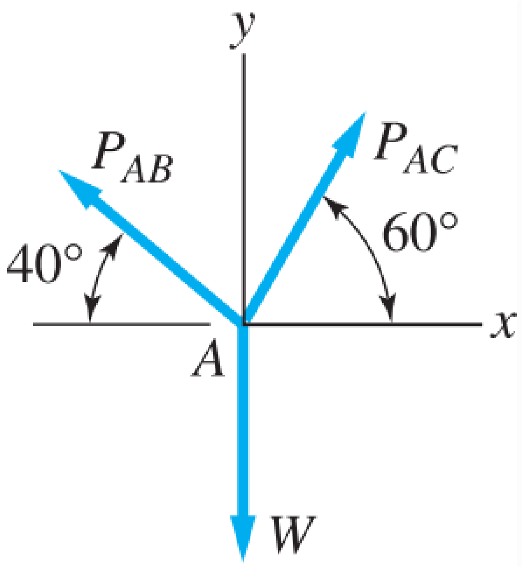
$$\sum F_y = 0 \quad +\uparrow \quad P_{AC}\sin 60^\circ + P_{AB}\sin 40^\circ - W = 0$$

Solving simultaneously, we get

$$P_{AB} = 0.5077W \qquad P_{AC} = 0.7779W$$



(a)



(b) FBD of pin A

Design for Normal Stress in Bar AB

The value of W that will cause the normal stress in bar AB to equal its working stress is given by

$$P_{AB} = (\sigma_w)_{AB}A_{AB}$$

$$0.5077W = (110 \times 10^6\ \mathrm{N/m^2})(800 \times 10^{-6}\ \mathrm{m^2})$$

$$W = 173.3 \times 10^3\ \mathrm{N}$$

Design for Normal Stress in Bar AC

The value of W that will cause the normal stress in bar AC to equal its working stress is found from

$$P_{AC} = (\sigma_w)_{AC} A_{AC}$$

$$0.7779\,W = (120 \times 10^6 \text{ N/m}^2)(400 \times 10^{-6} \text{ m}^2)$$

$$W = 61.7 \times 10^3 \text{ N} = 61.7 \text{ kN}$$

Choose the Correct Answer

The maximum safe value of W is the smaller of the preceding two values—namely,

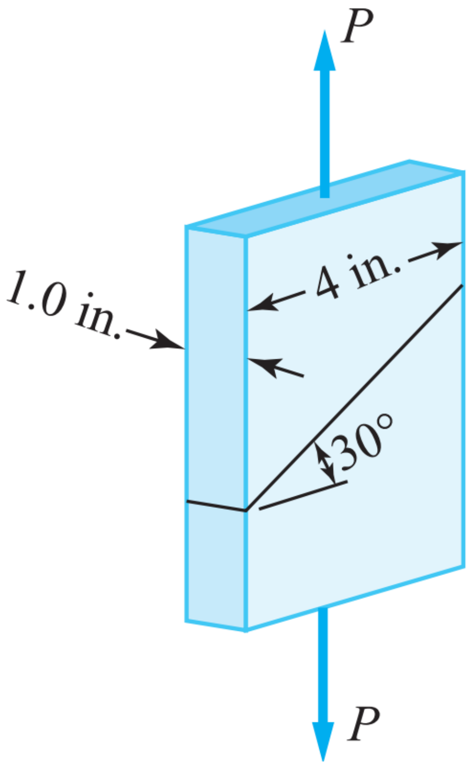
$W = 61.7\ kN$ Answer

We see that the stress in bar AC determines the safe value of W.

The other "solution," $W = 173.3\ kN$, must be discarded because it would cause the stress in AC to exceed its working stress of 120 MPa

Que.3. The rectangular wood panel is formed by gluing together two boards along the 30- degree seam as shown in the figure. Determine the largest axial force P that can be carried safely by the panel if the working stress for the wood is 1120 psi, and the normal and shear stresses in the glue are limited to 700 psi and 450 psi, respectively.

Sol: The most convenient method for analyzing this design-type problem is to calculate the largest safe value of P that satisfies each of the three design criteria. The smallest of these three values is the largest safe value of P for the panel.



Design for Working Stress in Wood

The value of P for which the wood would reach its working stress is found as follows:

$$\sigma = \frac{P}{A}\cos^2\theta$$

$$700 = \frac{P}{(4 \times 1.0)}\cos^2 30^\circ$$

$$P = 3730\text{lb}$$

Design for Shear Stress in Glue

The value of P that would cause the shear stress in the glue to equal its maximum value is computed as

$$\sigma = \frac{P}{2A} \sin 2\theta$$

$$450 = \frac{P}{2(4 \times 1.0)} \sin 60^\circ$$

$$P = 4160 \text{lb}$$

Choose the Correct Answer

Comparing the above three solutions, we see that the largest safe axial load that can be safely applied is governed by the normal stress in the glue, its value being

$P = 3730\ lb$ Answer

Shear Stress

- By definition, normal stress acting on an interior plane is directed perpendicular to that plane. Shear stress, on the other hand, is tangent to the plane on which it acts.
- Shear stress arises whenever the applied loads cause one section of a body to slide past its adjacent section. Three examples of shear stress are illustrated in Figure.
- Figure 1.11(a) shows two plates that are joined by a rivet. As seen in the FBD, the rivet must carry the shear force $V = P$. Because only one cross section of the rivet resists the shear, the rivet is said to be in single shear.
- The bolt of the clevis in Figure (b) carries the load P across two cross-sectional areas, the shear force being $V = \frac{P}{2}$ on each cross section. Therefore, the bolt is said to be in a state of double shear.
- In Figure(c) a circular slug is being punched out of a metal sheet. Here the shear force is P and the shear area is similar to the milled edge of a coin. The loads shown in Figure are sometimes referred to as direct shear to distinguish them from the induced shear as derived earlier.

The distribution of direct shear stress is usually complex and not easily determined. It is common practice to assume that the shear force V is uniformly distributed over the shear area A.

So, Shear stress calculated as

$$\tau = \frac{V}{A}$$

This shear stress must be interpreted as the average shear stress. It is often used in design to evaluate the strength of connectors, such as rivets, bolts, and welds.

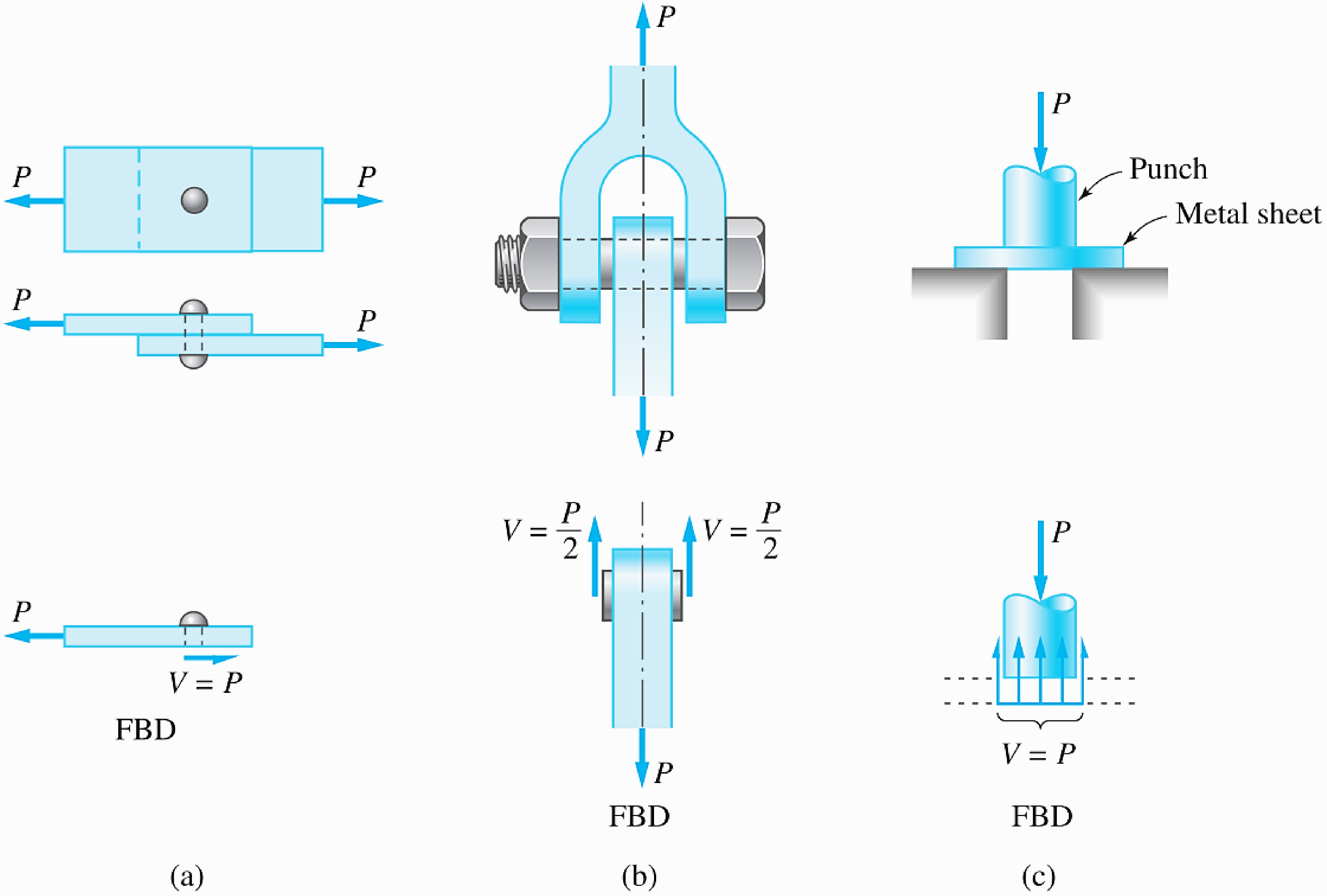


FIG. Examples of direct shear: (a) single shear in a rivet; (b) double shear in a bolt; and (c) shear in a metal sheet produced by a punch.

Bearing Stress

- If two bodies are pressed against each other, compressive forces are developed on the area of contact. The pressure caused by these surface loads is called bearing stress.
- Examples of bearing stress are the soil pressure beneath a pier and the contact pressure between a rivet and the side of its hole.
- If the bearing stress is large enough, it can locally crush the material, which in turn can lead to more serious problems. To reduce bearing stresses, engineers sometimes employ bearing plates, the purpose of which is to distribute the contact forces over a larger area.
- As an illustration of bearing stress, consider the lap joint formed by the two plates that are riveted together as shown in Figure (a). The bearing stress caused by the rivet is not constant; it actually varies from zero at the sides of the hole to a maximum behind the rivet as illustrated in Figure (b).
- The difficulty inherent in such a complicated stress distribution is avoided by the common practice of assuming that the bearing stress σ_b is uniformly distributed over a reduced area.

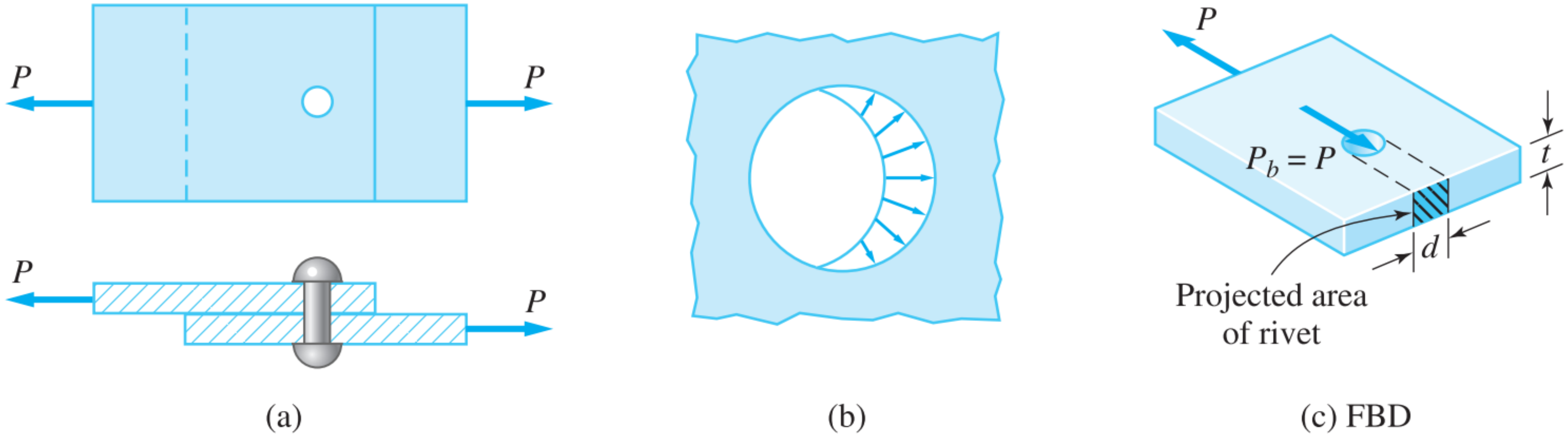


FIG. Example of bearing stress: (a) a rivet in a lap joint; (b) bearing stress is not constant; (c) bearing stress caused by the bearing force P_b is assumed to be uniform on projected area *td*.

- The reduced area A_b is taken to be the projected area of the rivet:

$$A_b = td$$

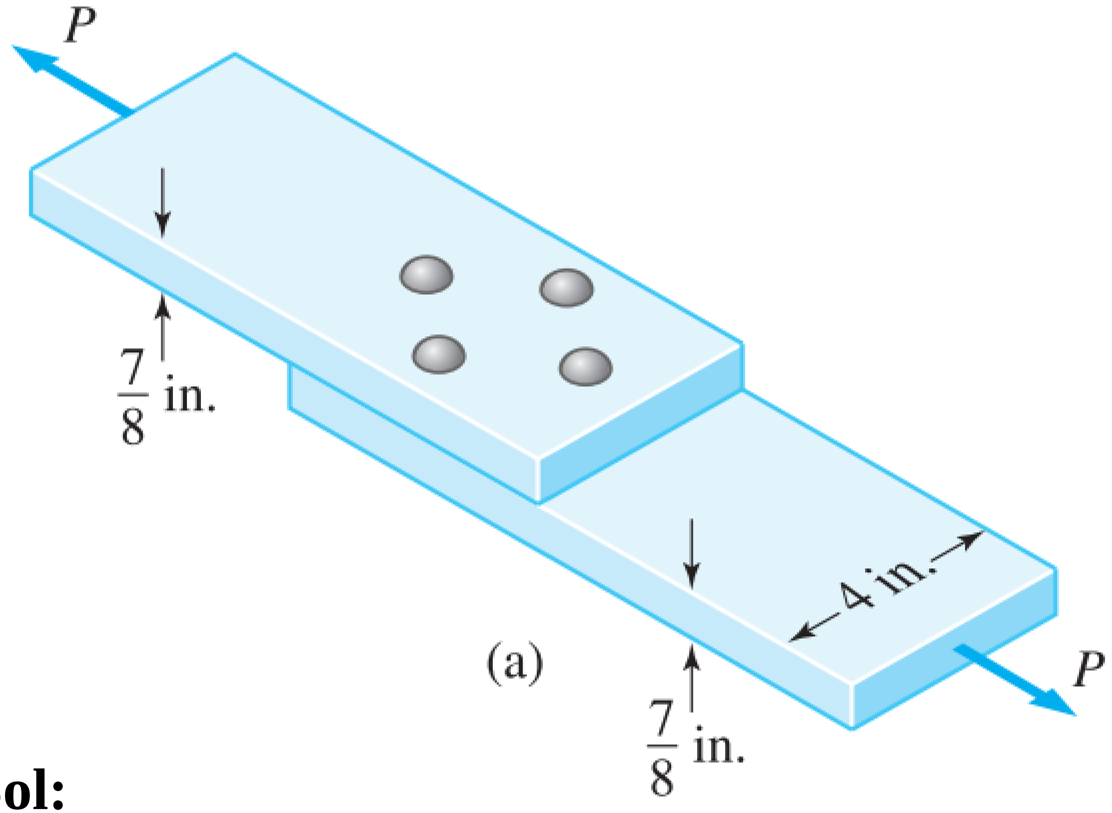
Diameter of the rivet

Thickness of the plate

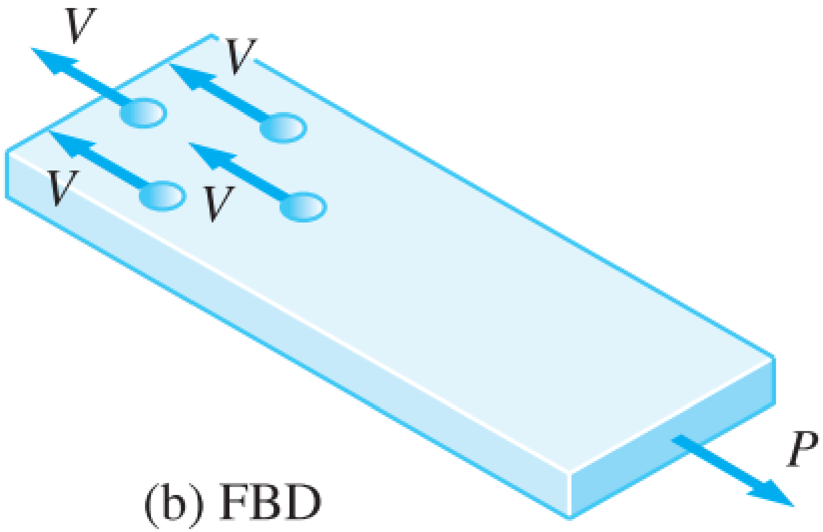
- From this FBD we see that the bearing force P_b equals the applied load P (the bearing load will be reduced if there is friction between the plates), so that the bearing stress becomes

$$\sigma_b = \frac{P_b}{A_b} = \frac{P}{td}$$

Que.4. The lap joint shown in Figure (a) is fastened by four rivets of 3/4-in. diameter. Find the maximum load P that can be applied if the working stresses are 14 ksi for shear in the rivet and 18 ksi for bearing in the plate. Assume that the applied load is distributed evenly among the four rivets, and neglect friction between the plates.



(a)



(b) FBD

Sol:

We will calculate P using each of the two design criteria. The largest safe load will be the smaller of the two values. Figure (b) shows the FBD of the lower plate. In this FBD, the lower halves of the rivets are in the plate, having been isolated from their top halves by a cutting plane.

Contd.

This cut exposes the shear forces V that act on the cross sections of the rivets. We see that the equilibrium condition is

$$V = \frac{P}{4}$$

Design for Shear Stress in Rivets

The value of P that would cause the shear stress in the rivets

to reach its working value is found as follows:

$$V = \tau A$$

$$\frac{P}{4} = (14 \times 10^3)\left[\frac{\pi(3/4)^2}{4}\right]$$

$$P = 24\,700 \text{ lb}$$

Contd.

Design for Bearing Stress in Plate

The shear force $V = \frac{P}{4}$ that acts on the cross section of one rivet is equal to the bearing force P_b due to the contact between the rivet and the plate. The value of P that would cause the bearing stress to equal its working value is computed as

$$P_b = \sigma_b t d$$

$$\frac{P}{4} = (18 \times 10^3)(7/8)(3/4)$$

$$P = 47\,300 \text{ lb}$$

Choose the Correct Answer

Comparing the above solutions, we conclude that the maximum safe load P that can be applied to the lap joint is

$$P = 24700\ lb \qquad \text{Answer}$$

with the shear stress in the rivets being the governing design criterion.

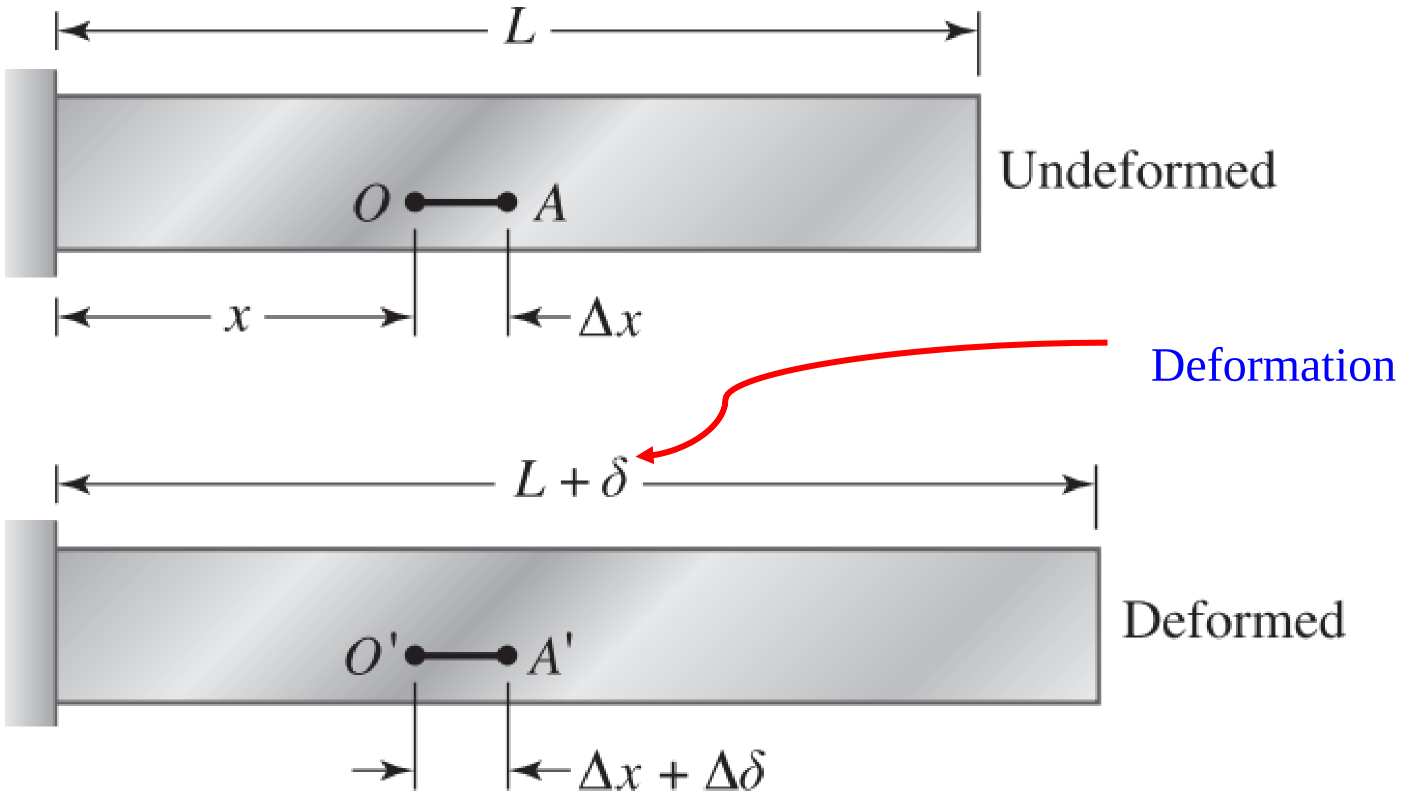
Strain

- So far, we have dealt mainly with the strength, or load-carrying capacity, of structural members.
- Here we begin our study of an equally important topic of mechanics of materials—deformations, or strains. In general terms, strain is a geometric quantity that measures the deformation of a body.
- There are two types of strain: normal strain, which characterizes dimensional changes, and shear strain, which describes distortion (changes in angles).
- Stress and strain are two fundamental concepts of mechanics of materials. Their relationship to each other defines the mechanical properties of a material, the knowledge of which is of the utmost importance in design.

Change in dimension of the specimen per unit original dimension is known as strain

- Normal strain or the linear strain (ε) (lowercase Greek epsilon) is defined as the elongation per unit length. Therefore, the normal strain in the bar in the axial direction, also known as the axial strain, is
- Lateral strain is defined as the change in diameter per unit diameter as in case of cylindrical specimen or change in lateral dimension per unit lateral dimension as in the case of a rectangular section.
- The elongation δ may be caused by an applied axial force, or an expansion due to an increase in temperature, or even a force and a temperature increase acting simultaneously. Strain describes the geometry of deformation, independent of what actually causes the deformation.

$$\epsilon = \frac{\delta}{L}$$



Deformation of a prismatic bar.

- Note that normal strain, being elongation per unit length, is a dimensionless quantity.
- If the bar deforms uniformly, then Equation represents the axial strain everywhere in the bar.
- Otherwise, this expression should be viewed as the average axial strain.

$$\epsilon = \lim_{\Delta x \to 0} \frac{\Delta \delta}{\Delta x} = \frac{d\delta}{dx}$$

$$\delta = \int_0^L d\delta = \int_0^L \epsilon \, dx$$

Mechanics of Materials

Chapter 3

Torsion

3.1 *Introduction*

- In many engineering applications, members are required to carry **torsional loads**.
- Consider the torsion of circular shafts. Because a circular cross section is an efficient shape for resisting torsional loads. Circular shafts are commonly used to transmit power in rotating machinery.
- Also discuss another important application—torsion of thin-walled tubes..

3.1 *Torsion of Circular Shafts*

a. Simplifying assumptions

- During the deformation, the cross sections are not distorted in any manner—they remain plane, and the radius r does not change. In addition, the length L of the shaft remains constant.

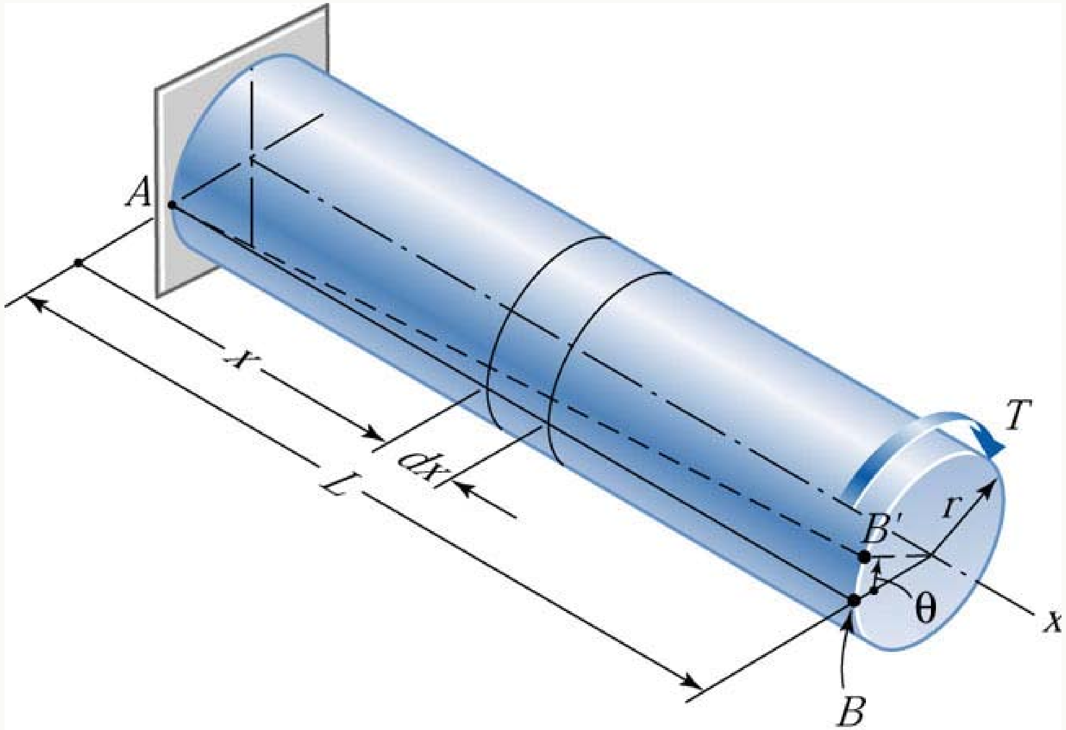
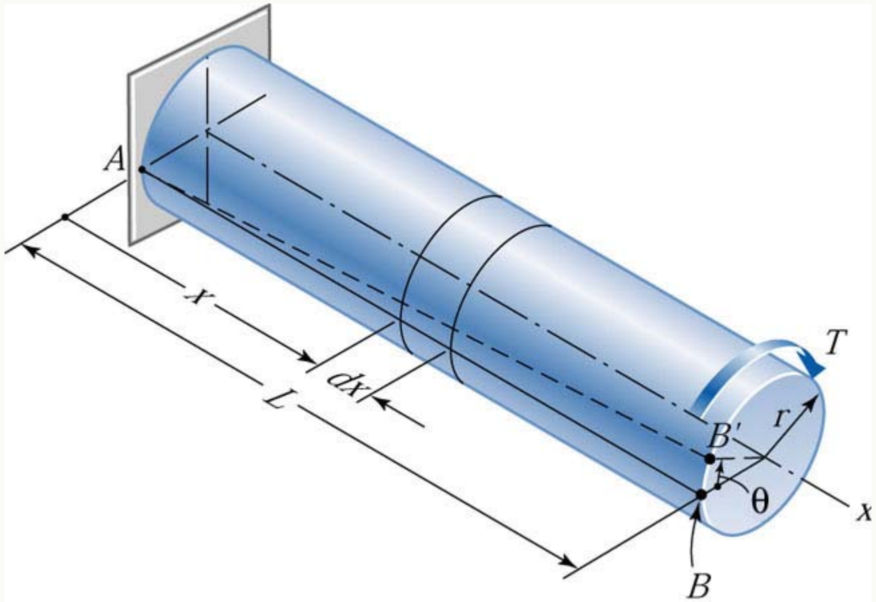


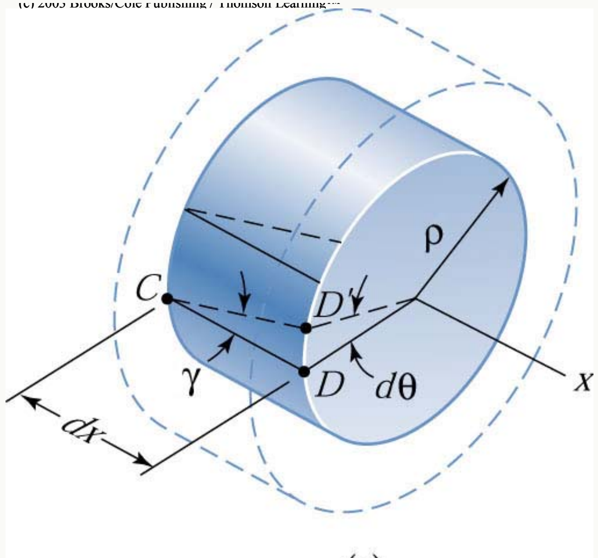
Figure 3.1 Deformation of a circular shaft caused by the torque *T*. The initially straight line *AB* deforms into a helix.

- Based on these observations, we make the following assumptions:
  - Circular cross sections remain plane (do not warp) and perpendicular to the axis of the shaft.
  - Cross sections do not deform (there is no strain in the plane of the cross section).
  - The distances between cross sections do not change (the axial normal strain is zero).
- *Each cross section rotates as a rigid entity about the axis of the shaft*. Although this conclusion is based on the observed deformation of a cylindrical shaft carrying a **constant** internal torque, we assume that the result remains valid even if the diameter of the shaft or the internal torque **varies** along the length of the shaft.

b. *Compatibility*

- Because the cross sections are separated by an infinitesimal distance, the difference in their rotations, denoted by the angle $d\theta$, is also infinitesimal.
- As the cross sections undergo the relative rotation $d\theta$, *CD* deforms into the helix *CD*. By observing the distortion of the shaded element, we recognize that the helix angle γ is the *shear strain* of the element.





(a)

From the geometry of Fig.3.2(a), we obtain $DD' = \rho\, d\theta = \gamma\, dx$, from which the shear strain γ is

$$\gamma = \frac{d\theta}{dx}\rho \qquad (3.1)$$

The quantity $d\theta/dx$ is the *angle of twist per unit length,* where θ is expressed in radians. The corresponding shear stress, illustrated in Fig. 3.2 (b), is determined from Hooke´s law:

$$\tau = G\gamma = G\frac{d\theta}{dx}\rho \qquad (3.2)$$

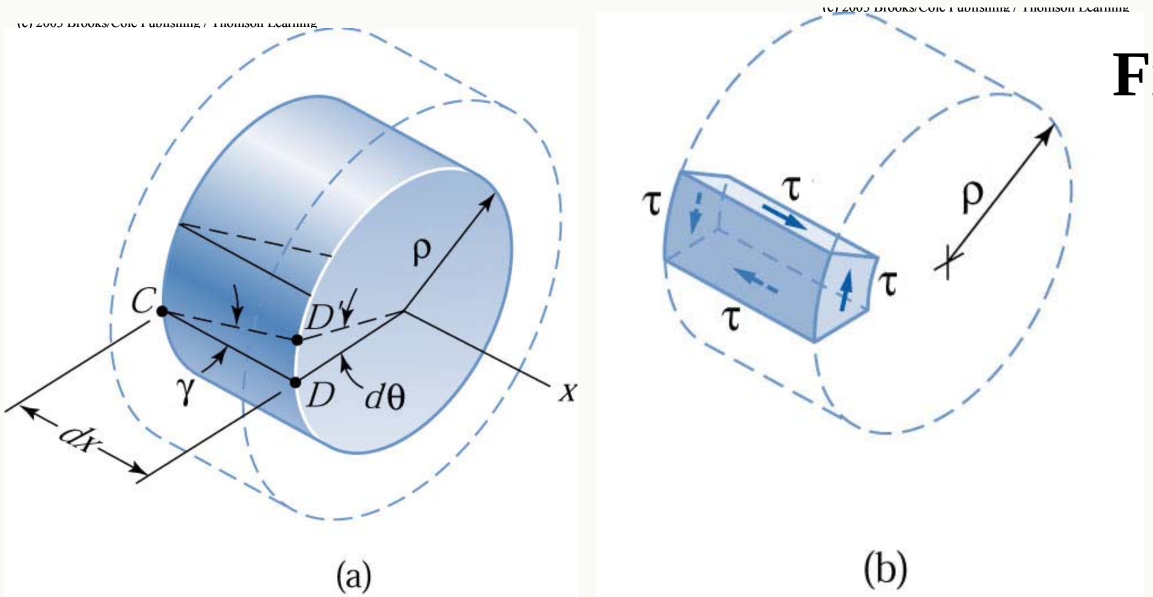


Figure 3.2 (a) Shear strain of a material element caused by twisting of the shaft; (b) the corresponding shear stress.

- *the shear stress varies linearly with the radial distance ρ from the axial of the shaft.* $\tau = G\gamma = G\frac{d\theta}{dx}\rho$
- The variation of the shear stress acting on the cross section is illustrated in Fig. 3.3. The maximum shear stress, denoted by τ_{max}, occurs at the surface of the shaft.
- Note that the above derivations assume neither a constant internal torque nor a constant cross section along the length of the shaft.

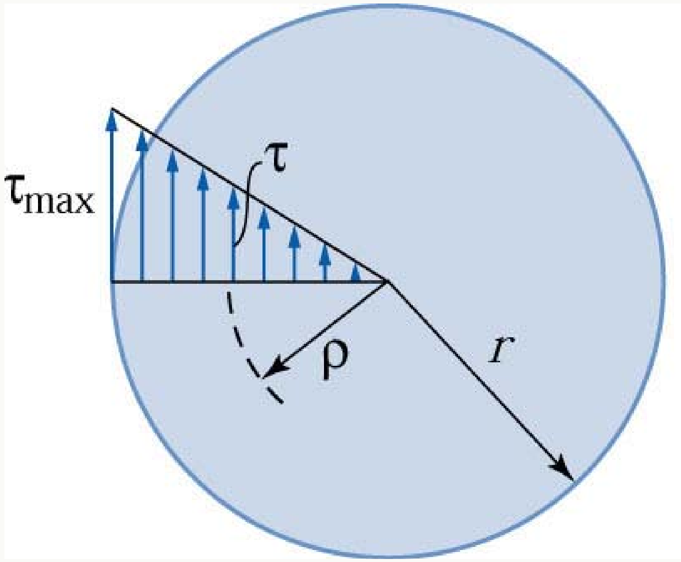


Figure 3.3 Distribution of shear stress along the radius of a circular shaft.

c. *Equilibrium*

- Fig. 3.4 shows a cross section of the shaft containing a differential element of area *dA* loaded at the radial distance ρ from the axis of the shaft.

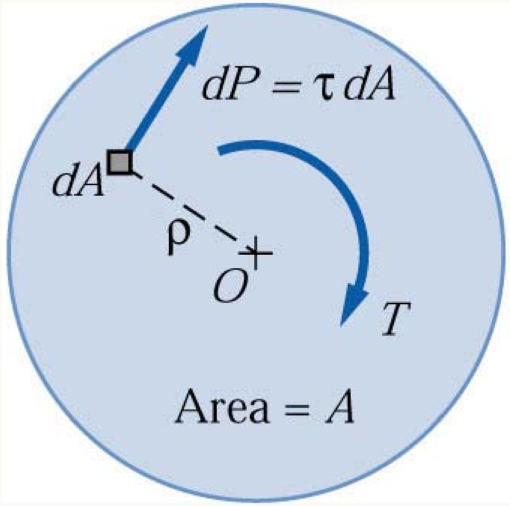


Figure 3.4 Calculating the Resultant of the shear stress acting on the cross section. Resultant is a couple equal to the internal torque *T*.

- The shear force acting on this area is $dP = \tau\, dA = G\,(d\theta/dx)\,\rho\, dA$, directed perpendicular to the radius. Hence, the moment (torque) of dP about the center O is $\rho\, dP = G\,(d\theta/dx)\,\rho^2 dA$. Summing the contributions and equating the result to the internal torque yields $\int_A \rho dP = T$, or

$$G\frac{d\theta}{dx}\int_A \rho^2 dA = T$$

Recognizing that is the polar moment of inertia of the cross-sectional area, we can write this equation as $G\,(d\,\theta\,/dx)\,J = T$, or

$$\frac{d\theta}{dx} = \frac{T}{GJ} \qquad (3.3)$$

The rotation of the cross section at the free end of the shaft, called **the angle of twist** θ , is obtained by integration:

$$\theta = \int_o^L d\theta = \int_o^L \frac{T}{GJ}dx \qquad (3.4a)$$

As in the case of a **prismatic bar** carrying a constant torque, then reduces **the torque-twist relationship**

$$\theta = \frac{TL}{GJ} \qquad (3.4b)$$

Note the similarity between Eqs. (3.4) and the corresponding formulas for axial deformation: $\delta = \int_o^L (P/EA)dx$ and $\delta = PL/(EA)$

Notes on the Computation of angle of Twist

- 1.In the U.S. Customary system, the consistent units are G [psi], T [$lb \cdot$ in] , and L [in.], and J [in^4]; in the SI system, the consistent units are G [Pa], T [$\text{N} \cdot \text{m}$], L [m], and J [m^4].
- 2.The unit of θ in Eqs. (3.4) is radians, regardless of which system of unit is used in the computation.
- 3.Represent torques as vectors using the right-hand rule, as illustrated in Fig. 3.5. The same sign convention applies to the angle of twist θ .

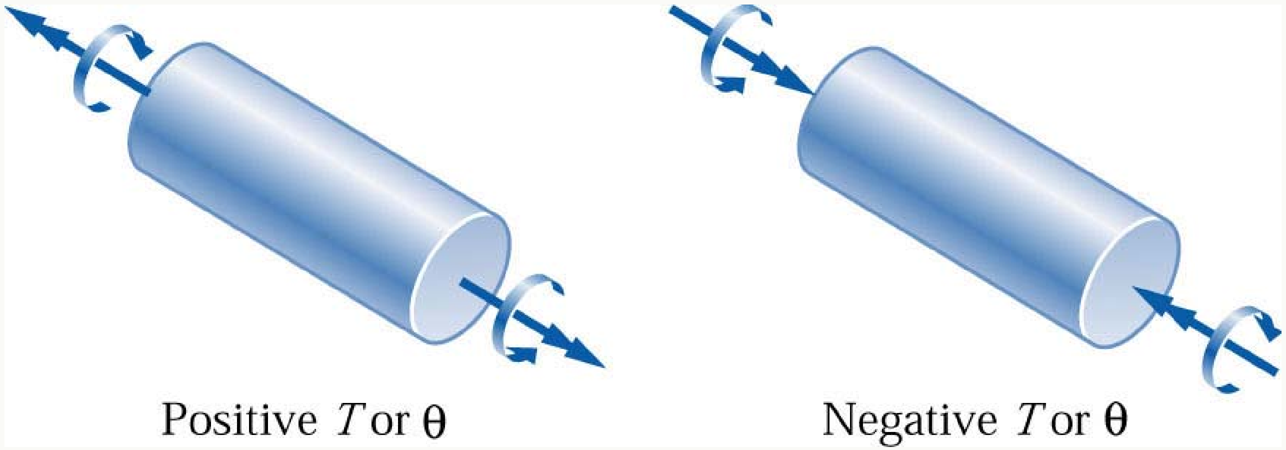


Figure 3.5 Sign Conventions for Torque *T* and angle of twist *T*.

d. *Torsion formulas*

- $G\,(d\theta/dx) = T/J$, which substitution into Eq. (3.2), $\tau = G\gamma = G\frac{d\theta}{dx}\rho$ gives the shear stress τ acting at the distance ρ from the center of the shaft, ***Torsion formulas*** :

$$\tau = \frac{T\rho}{J} \qquad (3.5a)$$

The **maximum** shear stress τ_{max} is found by replacing ρ by the radius r of the shaft:

$$\tau_{max} = \frac{Tr}{J} \qquad (3.5b)$$

- Because Hook´s law was used in the derivation of Eqs. (3.2)-(3.5), these formulas are valid if the shear stresses do not exceed the proportional limit of the material shear. Furthermore, these formulas are applicable only to **circular shafts**, either solid or hollow.

❑ The expressions for the polar moments of circular areas are :

Solid shaft : $$\tau_{\max} = \frac{2T}{\pi r^3} = \frac{16T}{\pi d^3} \qquad (3.5c)$$

Hollow shaft : $$\tau_{\max} = \frac{2TR}{\pi\left(R^4 - r^4\right)} = \frac{16TD}{\pi\left(D^4 - d^4\right)} \qquad (3.5d)$$

Equations (3.5c) and (3.5d) are called the ***torsion formulas***.

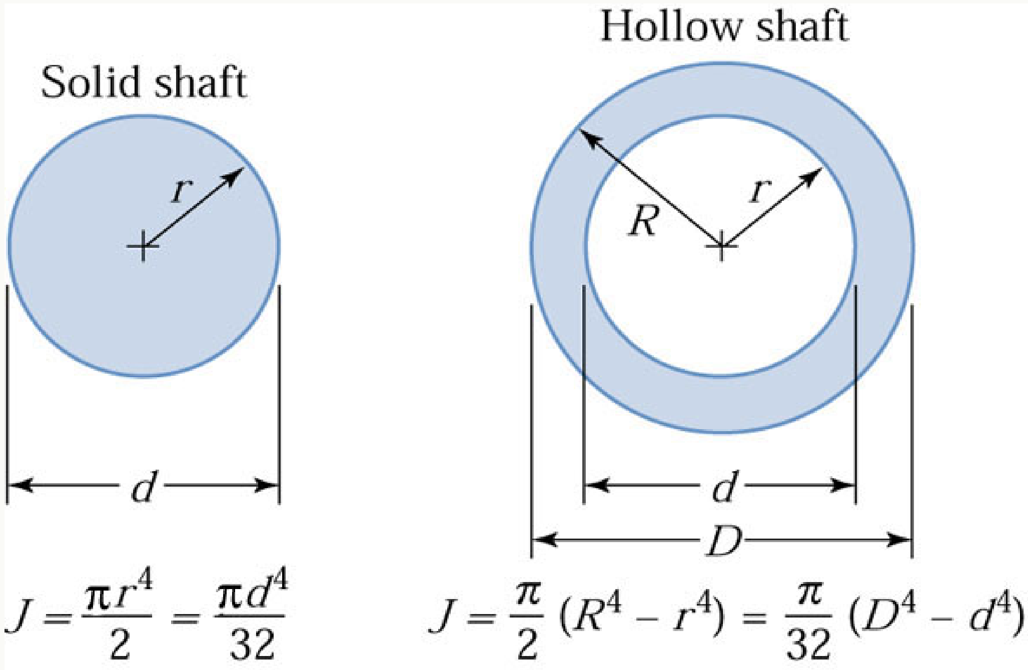


Figure 3.6 Polar moments of inertia of circular areas.

e. *Power transmission*

- Shafts are used to transmit power. The power ζ transmitted by a torque T rotating at the angular speed ω is given by $\zeta = T\omega$, where ω is measured in radians per unit time.
- If the shaft is rotating with a frequency of f revolutions per unit time, then $\omega = 2\pi f$, which gives $\zeta = T(2\pi f)$. Therefore, the torque can be expressed as

$$T = \frac{\zeta}{2\pi f} \qquad (3.6a)$$

- In SI units, ζ in usually measured in watts (1.0 W=1.0 N・m/s) and f in hertz (1.0 Hz = 1.0 rev/s); Eq. (3.6a) then determines the torque T in N・m.
- In U.S. Customary units with ζ in *lb*・in./s and f in hertz, Eq.(3.6a) calculates the torque T in *lb*・in.

- Because power in U.S. Customary units is often expressed in horsepower (1.0 hp = 550 *l*b • ft/s = 396×10³ *l*b • in./min), a convenient form of Eq.(3.6a) is

$$T(lb \cdot in) = \frac{\zeta(hp)}{2\pi f(rev/\min)} \times \frac{396 \times 10^3 (lb \cdot in./\min)}{1.0(hp)}$$

which simplifies to

$$T(lb \cdot in) = 63.0 \times 10^3 \frac{\zeta(hp)}{f(rev/\min)} \quad (3.6b)$$

f. *Statically indeterminate problems*

- Draw the required **free-body diagrams** and write the equations of **equilibrium**.
- Derive the **compatibility** equations from the restrictions imposed on the angles of twist.
- Use the **torque- twist relationships** in Eqs.(3.4) to express the angles of twist in the compatibility equations in terms of the torques.
- **Solve the equations** of equilibrium and compatibility for the torques.

***Sample Problem* 3.1**

A solid steel shaft in a rolling mill transmits 20 kW of power at 2 Hz. Determine the smallest safe diameter of the shaft if the shear stress τ_w is not to exceed 40 MPa and the angle of twist θ is limited to 6° in a length of 3 m. Use G = 83 GPa.

Solution

Applying Eq. (3.6a) to determine the torque:

$$T = \frac{P}{2\pi f} = \frac{20 \times 10^3}{2\pi(2)} = 1591.5 N \cdot m$$

To satisfy the strength condition, we apply the torsion formula, Eq. (3.5c):

$$\tau_{max} = \frac{Tr}{J} \qquad \tau_{max} = \frac{16T}{\pi d^3} \qquad 4 \times 10^6 = \frac{16(1591.5)}{\pi d^3}$$

Which yields $d = 58.7 \times 10^{-3}$ m = 58.7 mm.

Apply the torque-twist relationship, Eq. (3.4b), to determine the diameter necessary to satisfy the requirement of rigidity (remembering to convert θ from degrees to radians):

$$\theta = \frac{TL}{GJ} \qquad\qquad 6\left(\frac{\pi}{180}\right) = \frac{1591.5(3)}{\left(83 \times 10^9\right)\left(\pi d^4 / 32\right)}$$

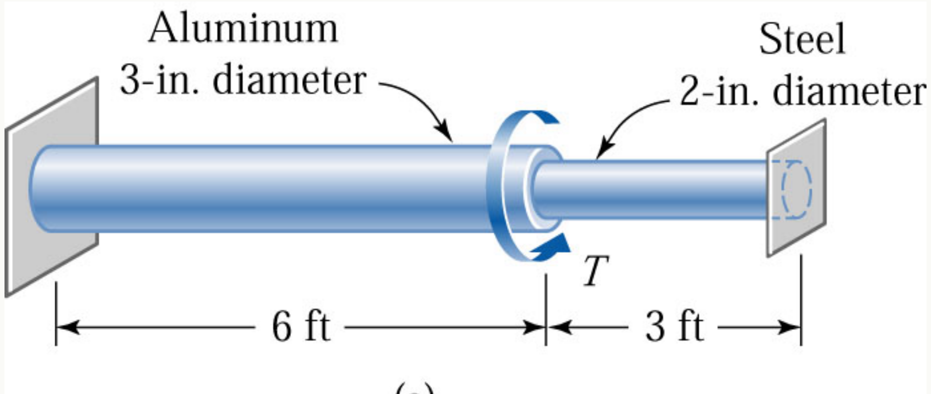
From which we obtain $d = 48.6 \times 10^{-3}$ m = 48.6 mm.

To satisfy both strength and rigidity requirements, we must choose the larger diameter-namely,

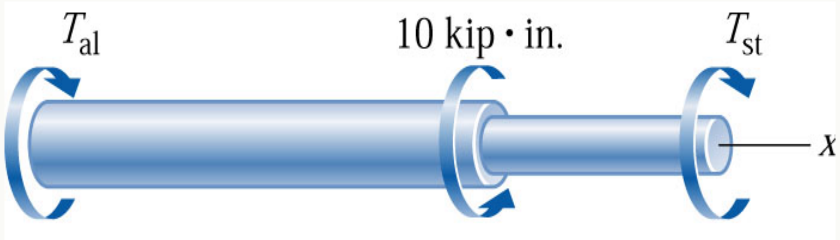
$d = 58.7$ mm. *Answer*

Sample problem 3.2

The shaft in Fig. (a) consists of a 3-in. -diameter aluminum segment that is rigidly joined to a 2-in. -diameter steel segment. The ends of the shaft are attached to rigid supports, Calculate the maximum shear stress developed in each segment when the torque T = 10 kip in. is applied. Use G = 4×10^6 psi for aluminum and G = 12×10^6 psi for steel.



(a)



(b) FBD

Solution

Equilibrium $\Sigma M_x = 0\,,\quad (10\times10^3) - T_{st} - T_{al} = 0 \qquad (a)$

This problem is statically indeterminate.

Compatibility the two segments must have the same angle of twist; that is, $\theta_{st} = \theta_{al}$ From Eq. (3.4b), this condition between.

$$\left(\frac{TL}{GJ}\right)_{st} = \left(\frac{TL}{GJ}\right)_{al} \qquad \frac{T_{st}(3\times 12)}{(12\times 10^6)\frac{\pi}{32}(2)^4} = \frac{T_{al}(6\times 12)}{(4\times 10^6)\frac{\pi}{32}(3)^4}$$

from which

$$T_{st} = 1.1852 \ T_{al} \qquad \text{(b)}$$

Solving Eqs. (a) and (b), we obtain

$$T_{al} = 4576 \ lb \cdot \text{in.} \qquad T_{st} = 5424 \ lb \cdot \text{in.}$$

the maximum shear stresses are

$$(\tau_{max})_{al} = \left(\frac{16T}{\pi d^3}\right)_{al} = \frac{16(4576)}{\pi(3)^3} = 863\,psi$$

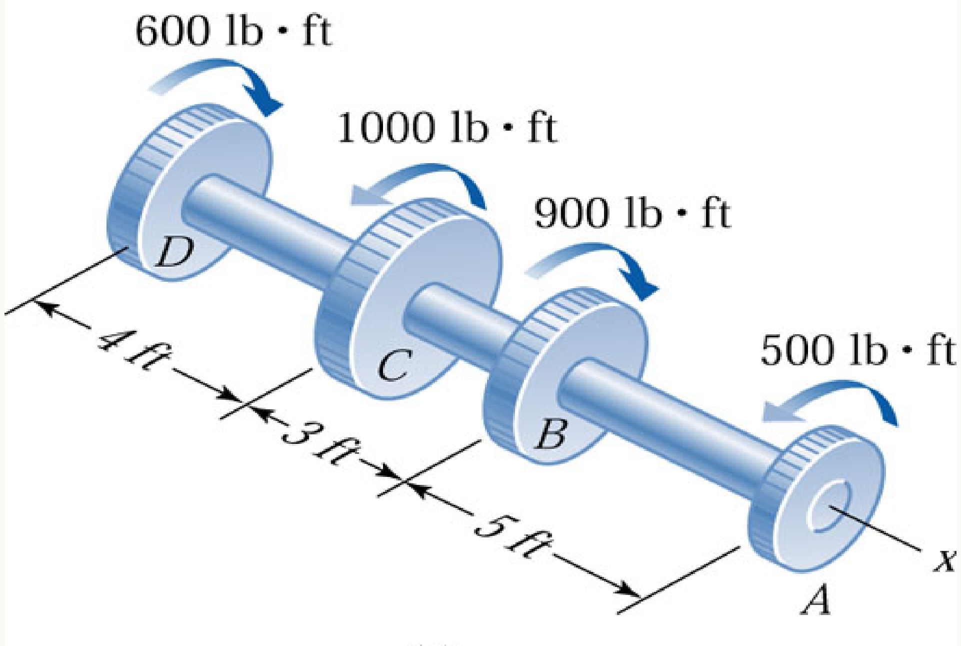
Answer

$$(\tau_{max})_{st} = \left(\frac{16T}{\pi d^3}\right)_{st} = \frac{16(5424)}{\pi(2)^3} = 3450\,psi$$

Answer

***Sample problem* 3.3**

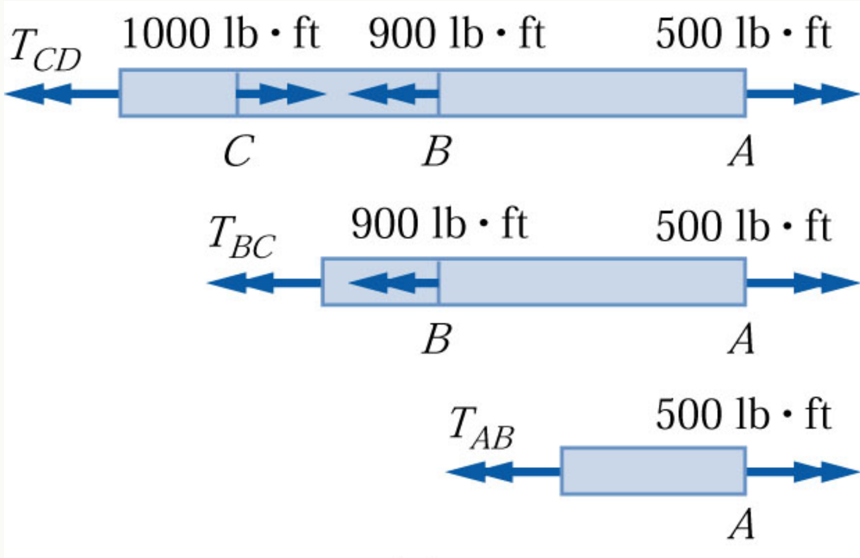
The four rigid gears, loaded as shown in Fig. (a), are attached to a 2-in.-diameter steel shaft. Compute the angle θ of rotation of gear A relative to gear *D*. Use $G = 12 \times 10^6$ psi for the shaft.



(a)

Solution

It is convenient to represent the torques as vectors (using the right-hand rule) on the FBDs in Fig. (b).



(b) FBDs

Solution

Assume that the internal torques T_{AB}, T_{BC}, and T_{CD} are positive according to the sign convention introduced earlier (positive torque vectors point away from the cross section). Applying the equilibrium condition $\sum M_x = 0$ to each FBD, we obtain

$$500 - 900 + 1000 - T_{CD} = 0$$

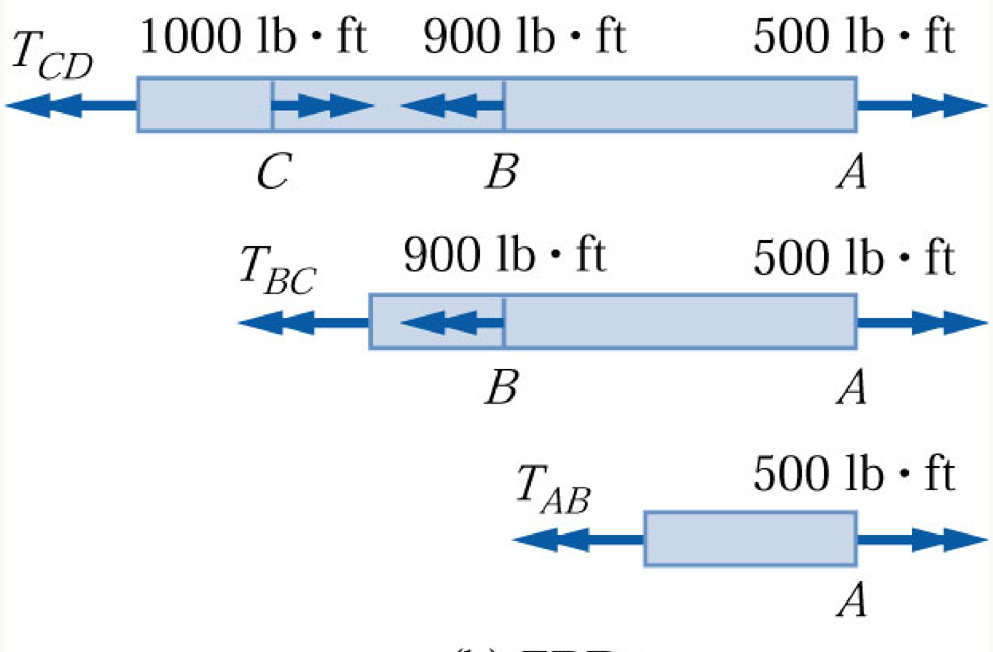
$$500 - 900 - T_{BC} = 0$$

$$500 - T_{AB} = 0$$

T_{AB} =500 $lb \cdot$ ft ,

T_{BC} =-400 $lb \cdot$ ft

T_{CD} = 600 $lb \cdot$ ft



(b) FBDs

The minus sign indicates that the sense of T_{BC} is opposite to that shown on the FBD.*A* is gear *D* were fixed.

This rotation is obtained by summing the angles of twist of the three segments:

$$\theta_{A/D} = \theta_{A/B} + \theta_{B/C} + \theta_{C/D}$$

Using Eq.(3.4b), we obtain (converting the lengths to inches and torques to pound-inches)

$$\theta_{A/D} = \frac{T_{AB}L_{AB} + T_{BC}L_{BC} + T_{CD}L_{CD}}{GJ}$$

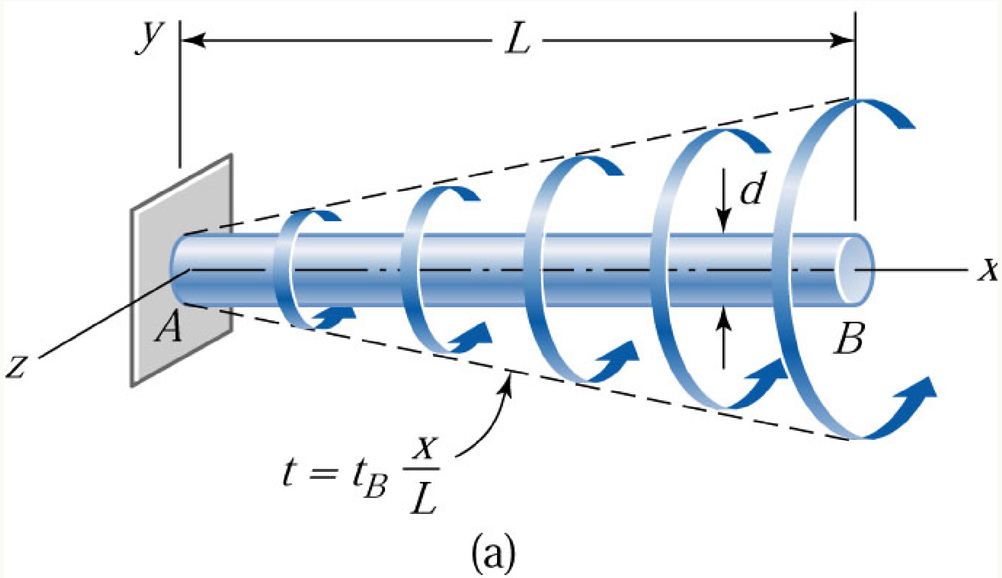
$$= \frac{(500\times12)(5\times12) - (400\times12)(3\times12) + (600\times12)(4\times12)}{\left[\pi(2)^4/32\right](12\times10)^6}$$

$$= 0.02827 \text{ rad} = 1.620^\circ \qquad \textit{Answer}$$

The positive result indicates that the rotation vector of *A* relative to *D* is in the positive *x*-direction: that is, θ_{AD} is directed counterclockwise when viewed from *A* toward *D*.

Sample Problem 3.4

Figure (a) shows a steel shaft of length $L = 1.5$ m and diameter $d = 25$ mm that carries a distributed toque of intensity (torque per unit length) $t = t_B(x/L)$, where $t_B = 200$ N· m/m. Determine (1) the maximum shear stress in the shaft; and (2) the angle of twist. Use $G = 80$ GPa for steel.



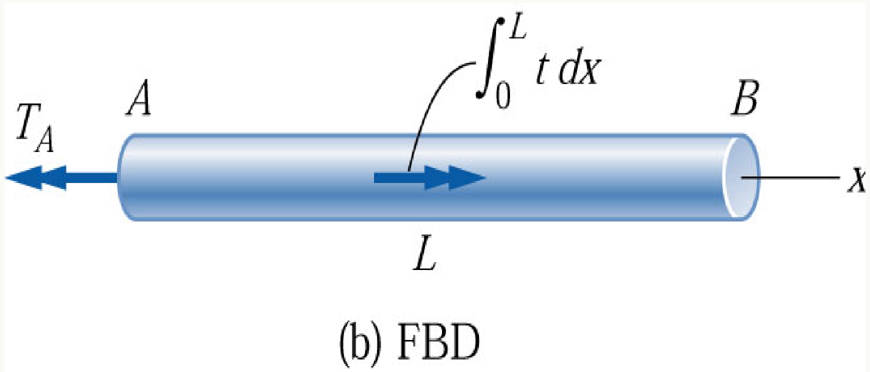
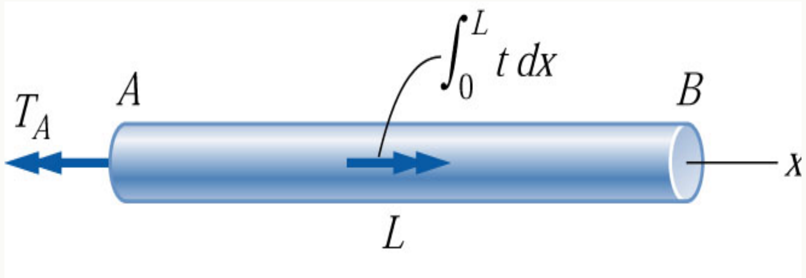


Figure (a) and (b) FBD

Solution



(b) FBD

Part 1

Figure (b) shows the FBD of the shaft. The total torque applied to the shaft is $\int_0^t tdx$. The maximum torque in the shaft is T_A, which occurs at the fixed support. From the FBD we get

$$\sum M_X = 0 \qquad \int_0^L tdx - T_A = 0$$

$$T_A = \int_0^L tdx = \int_0^L t_B \frac{x}{L} dx = \frac{t_B L}{2} = \frac{1}{2}(200)(1.5) = 150 N \cdot m$$

From Eq. (3.5c), the maximum stress in the shaft is

$$\tau_{\max} = \frac{16T_A}{\pi d^3} = \frac{16(150)}{\pi(0.025)^3} = 48.9 \times 10^6 \, Pa = 48.9 MPa \quad \textit{Answer}$$

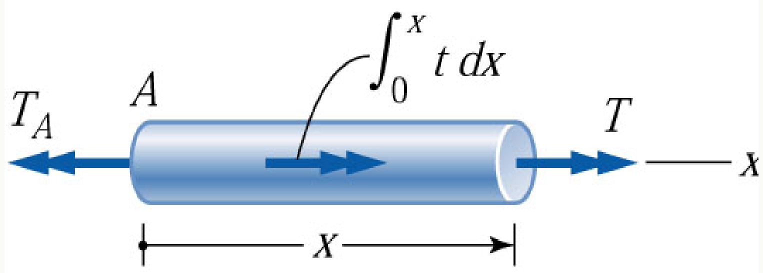
Part 2

The torque T acting on a cross section located at the distance x from the fixed end can be found from the FBD in Fig. (c):

$$\sum M_X = 0 \qquad T + \int_0^x t\,dx - T_A = 0$$

$$T = T_A - \int_0^x t\,dx = \frac{t_B L}{2} - \int_0^x t_B \frac{x}{L} dx$$

$$= \frac{t_B}{2L}\left(L^2 - x^2\right)$$



(c) FBD

From Eq. (3.4a), the angle θ of twist of the shaft is

$$\theta = \int_0^L \frac{T}{GJ} dx = \frac{t_B}{2LGJ}\int_0^L \left(L^2 - x^2\right)dx = \frac{t_B L^2}{3GJ}$$

$$= \frac{200(1.5)^2}{3\left(80\times 10^9\right)(\pi/32)(0.025)^4} = 0.0489 rad = 2.8^\circ \qquad Answer$$

3.3 *Torsion of Thin-Walled Tubes*

- Simple approximate formulas are available for thin-walled tubes. Such members are common in construction where light weight is of paramount importance.
- The tube to be prismatic (constant cross section), but the wall thickness t is allowed to vary within the cross section. The surface that lies midway between the inner and outer boundaries of the tube is called the ***middle surface***.

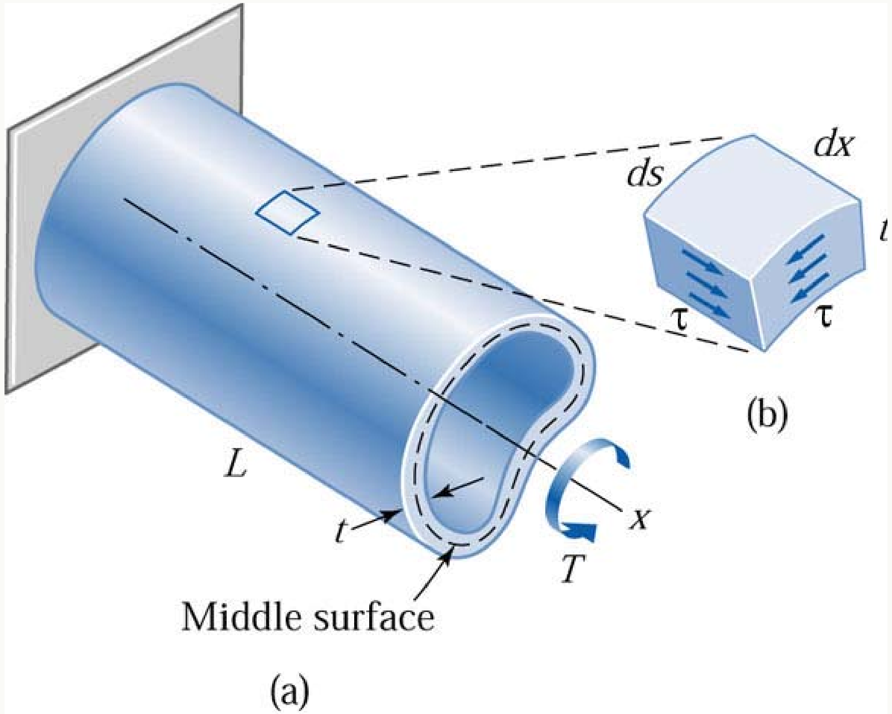
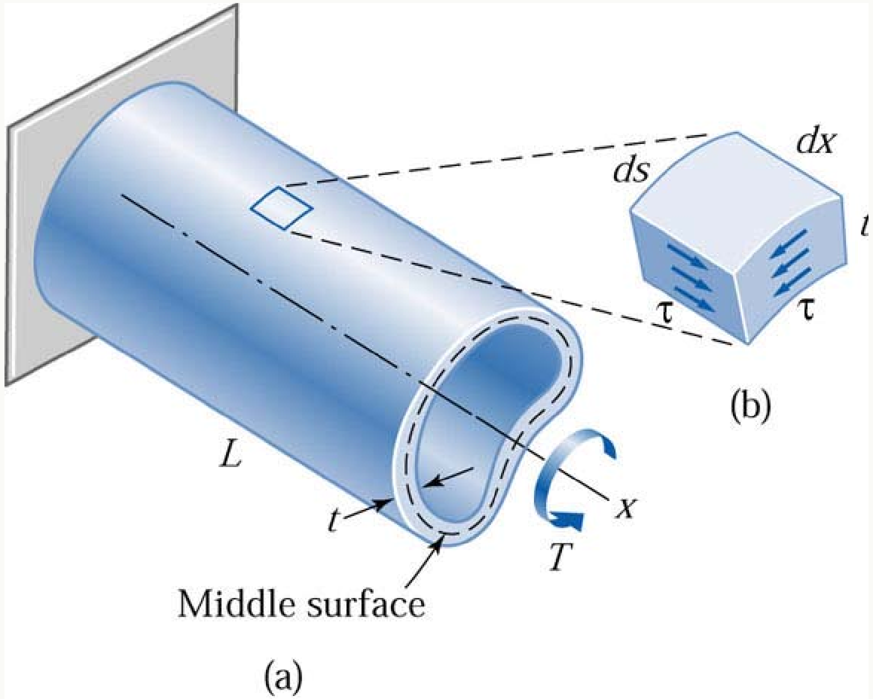


Figure 3.7 (a) Thin-walled tube in torsion; (b) shear stress in the wall of the tube.

- If thickness t is small compared to the overall dimensions of the cross section, the shear stress τ induced by torsion can be shown to be almost constant through the wall thickness of the tube and directed tangent to the middle surface, in Fig. (3.7b).
- At this time, it is convenient to introduce the concept of ***shear flow q***, defined as the shear force per unit edge length of the middle surface.

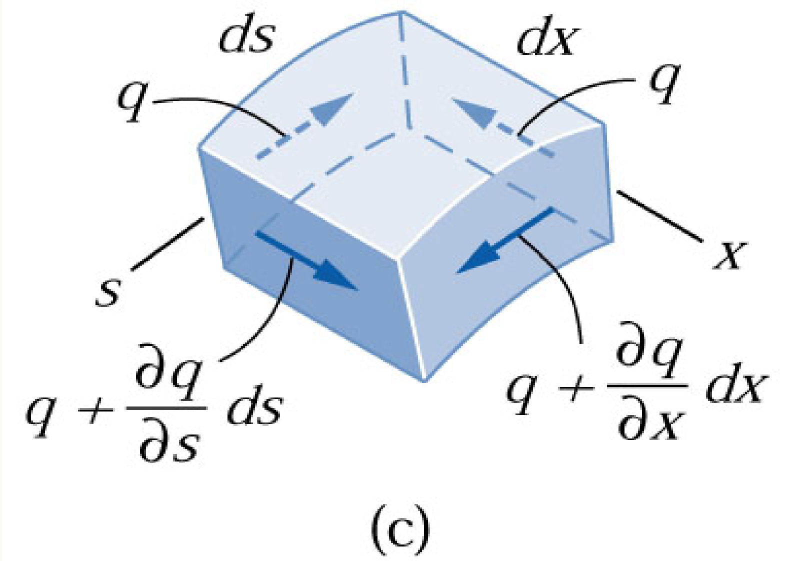


- the shear flow q is

$$q = \tau t \qquad (3.7)$$

If the shear stress is not constant through the wall thickness, then τ in Eq. (3.7) should be viewed as the average shear stress.

- *the shear flow is constant throughout the tube*. This result can be obtained by considering equilibrium of the element shown in Fig. 3.7(c).
- In labeling the shear flows, we assume that q varies in the longitudinal (x) as well as the circumferential (s) directions. The force acting on each side of the element is equal to the shear flow multiplied by the edge length, resulting in the equilibrium equations .



(c) Shear flows on wall element.

$$\sum F_x = 0 \quad \left(q + \frac{\partial q}{\partial s} ds\right) dx - q dx = 0$$

$$\sum F_s = 0 \quad \left(q + \frac{\partial q}{\partial x} dx\right) ds - q ds = 0$$

$\partial q / \partial x = \partial q / \partial s = 0$, thereby proving that the shear flow is constant throughout the tube.

- The shear force is $dP = qds$. The moment of the force about an arbitrary point 0 is $rdP = (qds)r$, where r is the perpendicular distance of *0* from the line of action of dP. The sum of these moment must be equal to the applied torque T; that is,

$$T = \oint_s qrds \qquad \text{(a)}$$

Which the integral is taken over the closed curve formed by the intersection of the middles surface and the cross section, called the ***median line***.

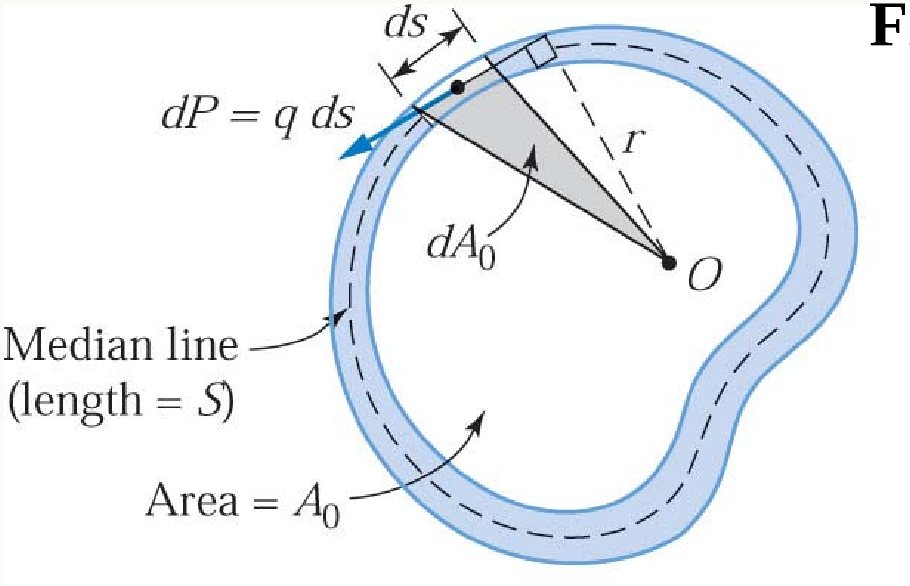
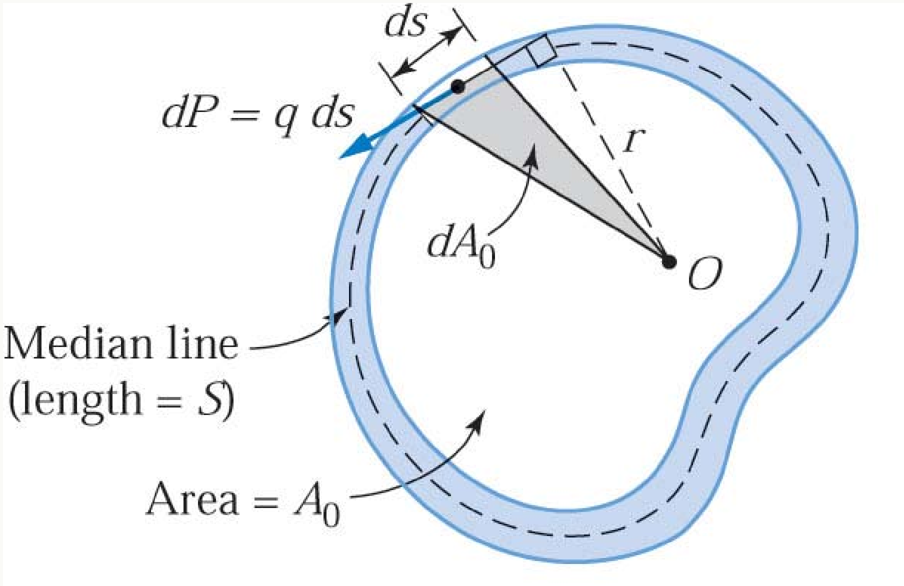


Figure 3.8 Calculating the resultant of the shear flow acting on the cross section of the tube. Resultant of a couple equal to the internal torque *T*.

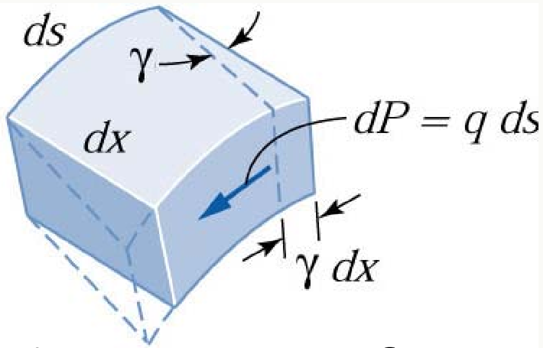
- But from Fig. 3.8 we see that $r\,ds = 2dA_o$, where dA_o is the area of the shaded triangle. Therefore, $\oint_s r\,ds = 2A_0$, where A_o is the area of the cross section that is enclosed by the median line.



$$T = \oint_s qr\,ds \qquad T = 2A_0 q \qquad (3.8a)$$

- from which the shear flow is

$$q = \frac{T}{2A_0} \qquad (3.8b)$$



- Determining the work done by the shear flow acting on the element in Fig. 3.7(c).

$$dU = \frac{1}{2}\left(force \times dis\tan ce\right) = \frac{1}{2}(qds)(\gamma dx)$$

Figure 3.9 Deformation of element caused by shear flow.

Substituting $\gamma = \tau / G = q / Gt$ yields

$$dU = \frac{1}{2}(qds)(\gamma dx) \qquad dU = \frac{q^2}{2Gt} dsdx \tag{b}$$

The work *U* of the shear flow for the entire tube is obtained by integrating Eq. (b) over the middle surface of the tube. Noting that q and *G* are constants and t is independent of *x*,

$$U = \frac{q^2}{2G}\int_0^L \left(\oint_s \frac{ds}{t} \right) dx = \frac{q^2 L}{2G}\oint_s \frac{ds}{t} \tag{c}$$

Conservation of energy requires *U* to be equal to the work of the applied torque that is, $U = T\theta/2$. After substituting the expression for q from Eq. (3.8b) into Eq. (c),

$$q = \frac{T}{2A_0} \qquad \left(\frac{T}{2A_0} \right)^2 \frac{L}{2G}\oint_s \frac{ds}{t} = \frac{1}{2} T\theta$$

The angle of twist of the tube is

$$\theta = \frac{TL}{4GA_0^2}\oint_s \frac{ds}{t} \qquad (3.9a)$$

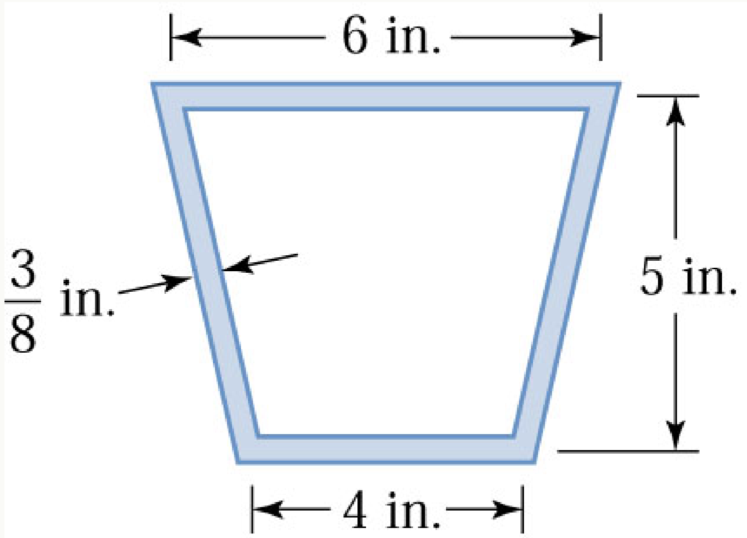
If t is constant, we have $\oint_s (ds/t) = S/t$, where S is the length of the median line Therefore, Eq. (3.9a) becomes

$$\theta = \frac{TLS}{4GA_0^2 t} \qquad (\text{ constant } t) \qquad (3.9b)$$

- If the tube is not cylindrical, its cross sections do not remain plane but tend to warp. Tube with very thin walls can fail by bucking which the stresses are still within their elastic ranges. steel tubes of circular cross section require $r/t < 50$ to forestall buckling due to torsion.
- Shape re-entrant corners in the cross section of the tube should also be avoided because they cause stress concentration. The shear stress at the inside boundary of a corner can be considerably higher than the average stress.

Sample Problem 3.5

A steel tube with the cross section shown carries a torque T. The tube is 6 ft long and has a constant wall thickness of 3/8 in. (1) Compute the torsional stiffness $k = T/\theta$ of the tube. (2) If the tube is twisted through 0.5°, determine the shear stress in the wall of the tube. Use $G = 12 \times 106$ psi. and neglect stress concentrations at the corners.



3.5 Figure (a)

Solution

Part1

Because the wall thickness is constant, the angle of twist is given by Eq. (3.9b):

$$\theta = \frac{TLS}{4GA_0^2 t}$$

Therefore, the torsional stiffness of the tube can be computed from

$$\kappa = \frac{T}{\theta} = \frac{4GA_0^2 t}{LS}$$

The area enclosed by the median line is

$$A_0 = averagewit\ h \times height = \left(\frac{6+4}{2}\right)(5) = 25 in.^2$$

And the length of the median line is

$$S = 6 + 4 + 2\sqrt{1^2 + 5^2} = 20.20 in.$$

Consequently, the torsional stiffness becomes

$$\kappa = \frac{4\left(12\times10^6\right)(25)^2(3/8)}{(6\times12)(20.20)} = 7.735\times10^6\ lb\cdot in./rad$$

$$= 135.0 \times 10^3 \text{ lb} \cdot \text{in./deg} \qquad \textit{Answer}$$

Part 2

The toque required to produce an angle of twist of 0.5° is

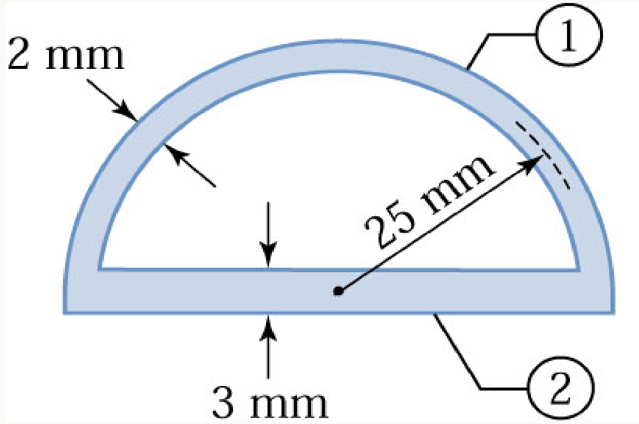
$$T = k\theta = (135.0 \times 10^3)(0.5) = 67.5 \times 10^3 \text{ lb} \cdot \text{in.}$$

which results in the shear flow $q = \frac{T}{2A_0} = \frac{67.5\times10^3}{2(25)} = 1350 lb/in.$

The corresponding shear stress is $\tau = \frac{q}{t} = \frac{1350}{3/8} = 3600\, psi$ Answer

Sample Problem 3.6

An aluminum tube, 1.2 m long, has the semicircular cross section shown in the figure. If stress concentrations at the corners are neglected, determine (1) the torque that causes a maximum shear stress of 40MPa.and (2) the corresponding angle of twist of the tube. Use G = 28 GPa for aluminum.



Solution

the shear flow that causes a maximum shear stress of 40 MPa is

$$q = \tau t = (40 \times 10^6)(0.002) = 80 \times 10^3 \text{ N/m}$$

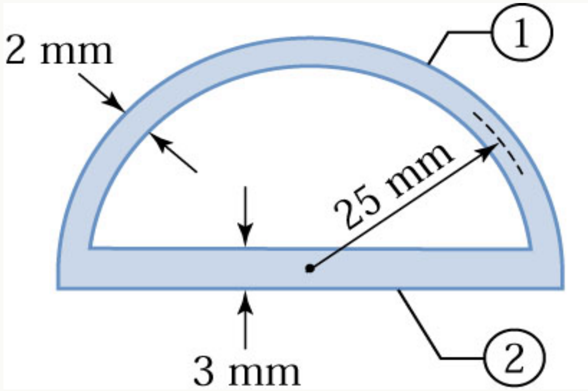
The cross-sectional area enclosed by the median line is

$$A_0 = \frac{\pi r^2}{2} = \frac{\pi (0.25)^2}{2} = 0.9817 \times 10^{-3} m^2$$

$$T = 2A_0 q = 2(0.9817\times10^{-3})(80\times10^{3}) = 157.07\, N\cdot m$$

Part 2

The cross section consists of two parts, labeled 1 and 2 in the figure, each having a constant thickness.



$$\oint_s \frac{ds}{t} = \frac{1}{t_1}\int_{s1} ds + \frac{1}{t_2}\int_{s2} ds = \frac{S_1}{t_1} + \frac{S_2}{t_2}$$

where S_1 and S_2 are the lengths of parts 1 and 2, respectively.

$$\oint_s \frac{ds}{t} = \frac{\pi\gamma}{t_1} + \frac{2\gamma}{t_2} = \frac{\pi(25)}{2} + \frac{2(25)}{3} = 55.94$$

and Eq. (3.9a) yields for the angle of twist

$$\theta = \frac{TL}{4GA_0^2}\oint_s \frac{ds}{t} = \frac{157.07(1.2)}{4(28\times10^{9})(09817\times10^{-3})^2}(55.94)$$

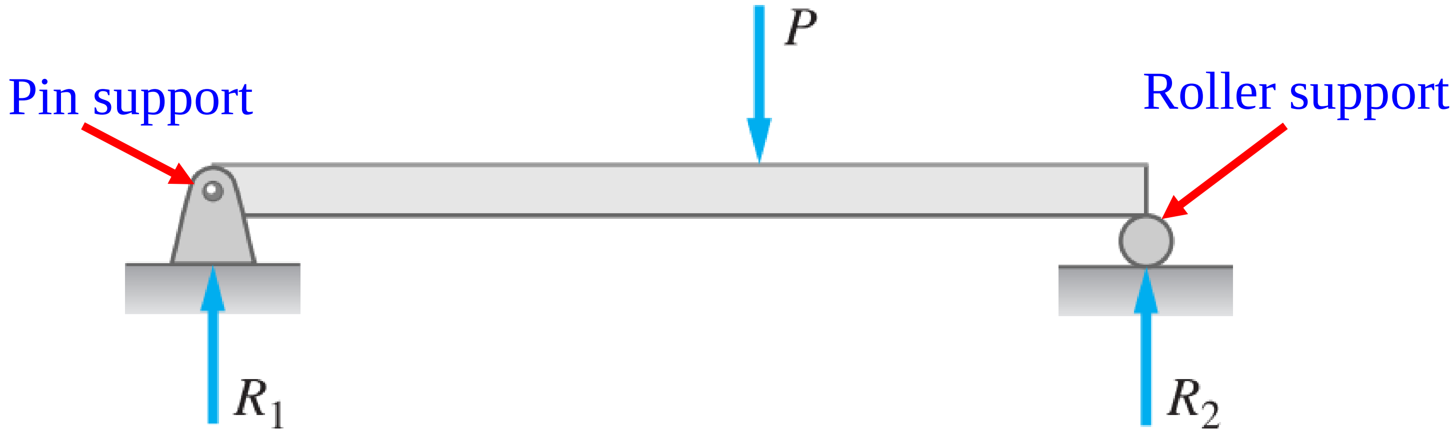
$$= 0.0977\, rad = 5.60^\circ \qquad Answer$$

Shear Force and Bending Moment Diagram

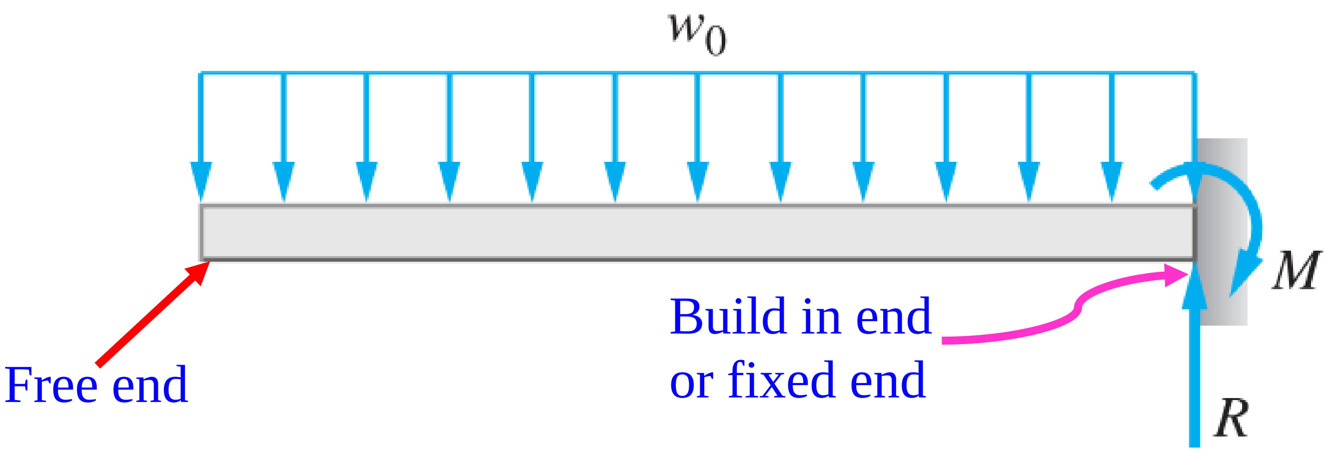
- The term beam refers to a slender bar that carries transverse loading; that is, the applied forces are perpendicular to the bar.
- In a beam, the internal force system consists of a shear force and a bending moment acting on the cross section of the bar.
- As we have seen in previous chapters, axial and torsional loads often result in internal forces that are constant in the bar, or over portions of the bar.
- The study of beams, however, is complicated by the fact that the shear force and the bending moment usually vary continuously along the length of the beam.
- The internal forces give rise to two kinds of stresses on a transverse section of a beam: (1) normal stress that is caused by the bending moment and

  (2) shear stress due to the shear force.

Support and loads



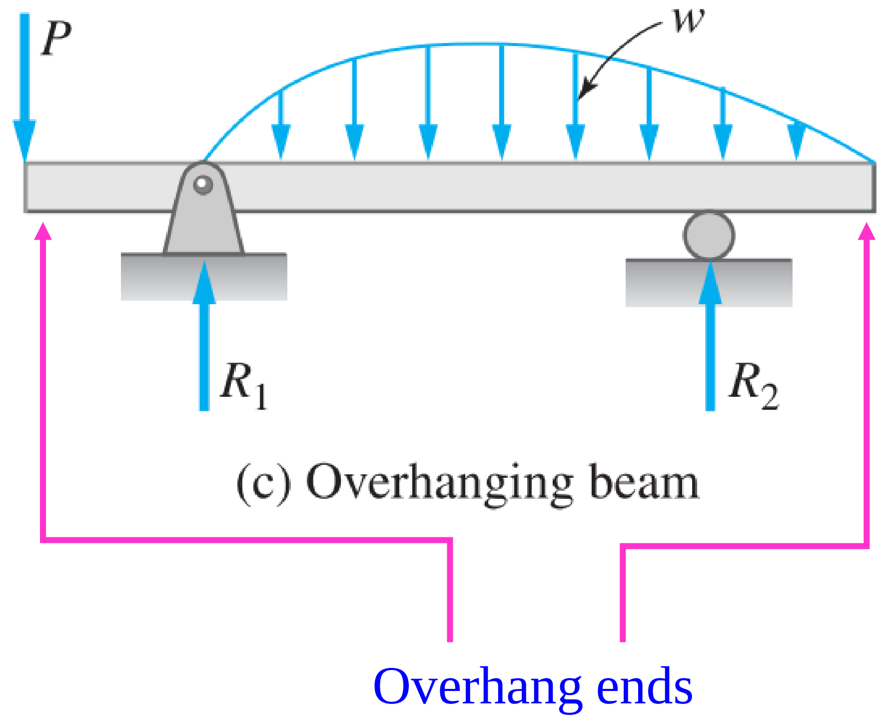
(a) Simply supported beam



(b) Cantilever beam

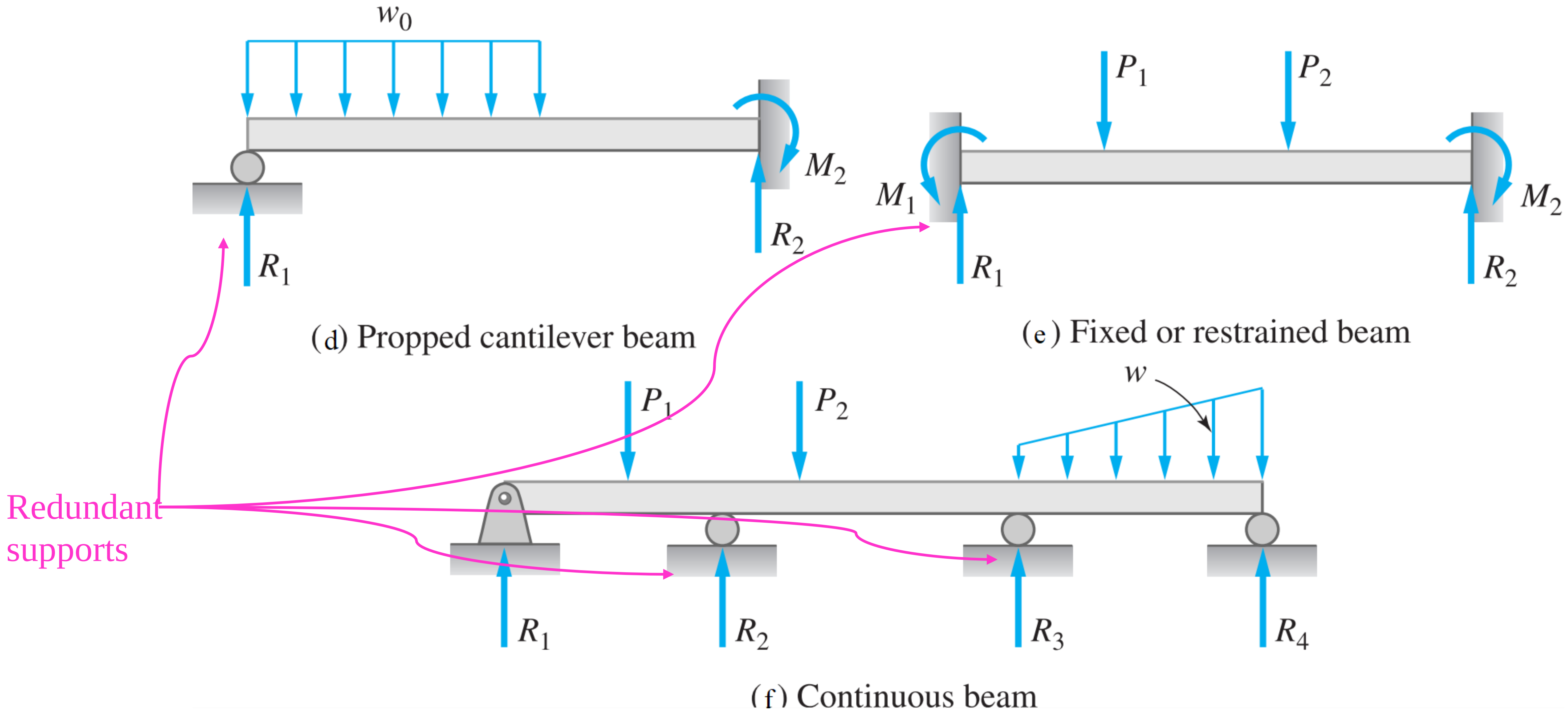
The pin support prevents displacement of the end of the beam, but not its rotation. The term roller support refers to a pin connection that is free to move parallel to the axis of the beam; hence, this type of support suppresses only the transverse displacement.

The built-in support prevents displacements as well as rotations of the end of the beam.



- An overhanging beam, illustrated in Figure(c), is supported by a pin and a roller support, with one or both ends of the beam extending beyond the supports. The three types of beams are statically determinate because the support reactions can be found from the equilibrium.
- A concentrated load, such as P in Figure (a), is an approximation of a force that acts over a very small area.
- In contrast, a distributed load is applied over a finite area. If the distributed load acts on a very narrow area, the load may be approximated by a line load. The intensity w of this loading is expressed as force per unit length (lb/ft, N/m, etc.).
- The load distribution may be uniform, as shown in Figure(b), or it may vary with distance along the beam, as in Figure(c).
- The weight of the beam is an example of distributed loading, but its magnitude is usually small compared to the loads applied to the beam.

Figures (d), (e), and (f) show other types of beams. These beams are over-supported in the sense that each beam has at least one more reaction than is necessary for support. Such beams are statically indeterminate; the presence of these redundant supports requires the use of additional equations obtained by considering the deformation of the beam.

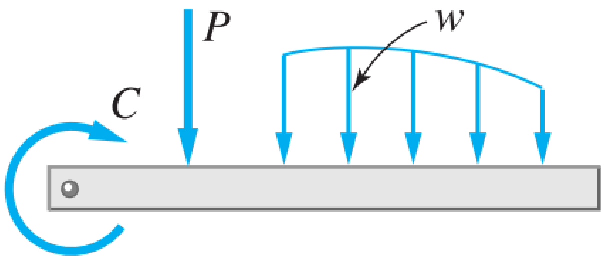
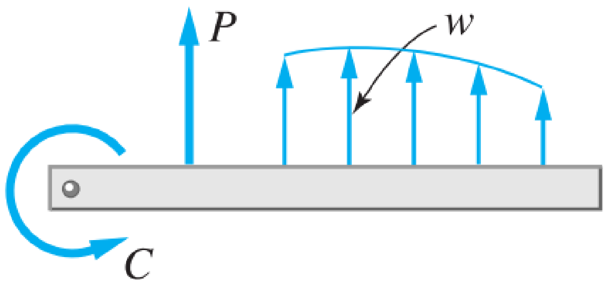
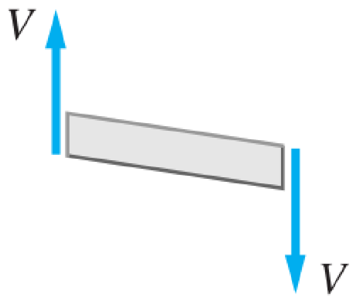
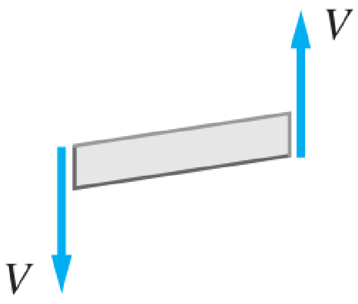
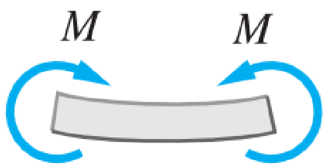
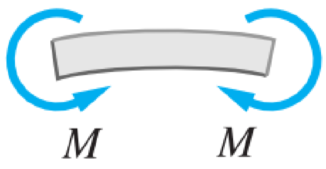


Shear force and bending moment diagram

- The determination of the internal force system acting at a given section of a beam is straightforward: We draw a free-body diagram that exposes these forces and then compute the forces using equilibrium equations.
- However, the goal of beam analysis is more involved—we want to determine the shear force V and the bending moment M at every cross section of the beam.
- To accomplish this task, we must derive the expressions for V and M in terms of the distance x measured along the beam. By plotting these expressions to scale, we obtain the shear force and bending moment diagrams for the beam.
- The shear force and bending moment diagrams are convenient visual references to the internal forces in a beam; in particular, they identify the maximum values of V and M.

Sign conventions

- For consistency, it is necessary to adopt sign conventions for applied loading, shear forces, and bending moments.
- External forces that are directed downward; external couples that are directed clockwise taken as positive.
- Shear forces that tend to rotate a beam element clockwise taken as positive.

	Positive	Negative
External loads		
Shear force		
Bending moment		

- Bending moments that tend to bend a beam element concave upward (the beam "smiles") known as sagging considered as positive, whereas bending moments that tend to bend a beam element concave downward (the beam "sad") known as hogging considered as negative.

The following is a general procedure for obtaining shear force and bending moment diagrams of a statically determinate beam: .

- Compute the support reactions from the FBD of the entire beam.
- Divide the beam into segments so that the loading within each segment is continuous. Thus, the end-points of the segments are discontinuities of loading, including concentrated loads and couples.

Perform the following steps for each segment of the beam:

- Introduce an imaginary cutting plane within the segment, located at a distance x from the left end of the beam, that cuts the beam into two parts.
- Draw a FBD for the part of the beam lying either to the left or to the right of the cutting plane, whichever is more convenient. At the cut section, show V and M acting in their positive directions.

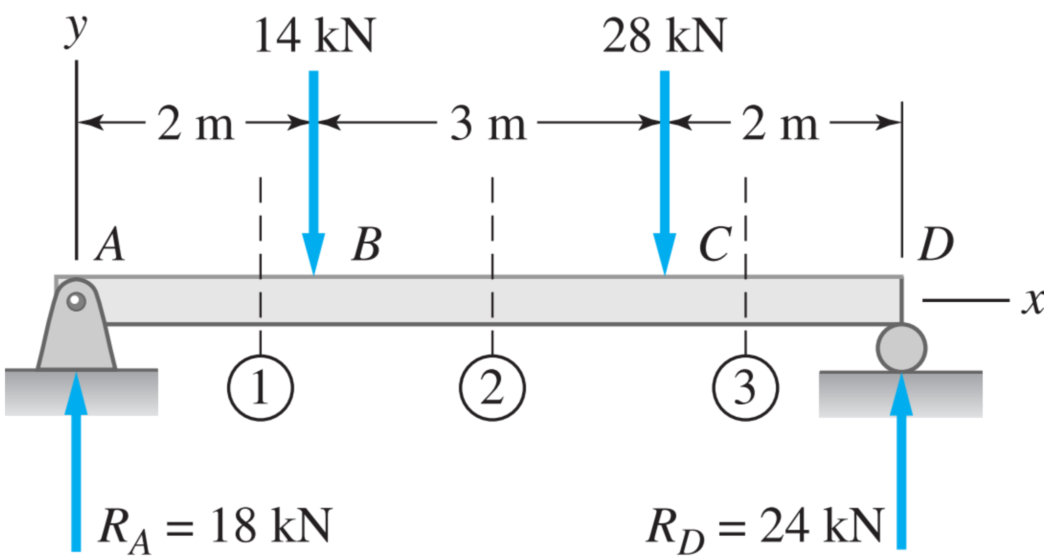
- Determine the expressions for V and M from the equilibrium equations obtainable from the FBD. These expressions, which are usually functions of x, are the shear force and bending moment equations for the segment.
- Plot the expressions for V and M for the segment. It is visually desirable to draw the V-diagram below the FBD of the entire beam, and then draw the M-diagram below the V-diagram.

The bending moment and shear force diagrams of the beam are composites of the V- and M-diagrams of the segments. These diagrams are usually discontinuous and/or have discontinuous slopes at the end-points of the segments due to discontinuities in loading.

Case-I: The simply supported beam in Fig. (a) carries two concentrated loads. (1) Derive the expressions for the shear force and the bending moment for each segment of the beam. (2) Draw the shear force and bending moment diagrams. Neglect the weight of the beam. Note that the support reactions at A and D have been computed and are shown in Figure.

Segment AB $(0 < x < 2m)$

Figure (b) shows the FBDs for the two parts of the beam that are separated by section 1 , located within segment AB. Note that we show V and M acting in their positive directions according to the sign conventions. Because V and M are equal in magnitude and oppositely directed on the two FBDs, they can be computed using either FBD. The analysis of the FBD of the part to the left of section 1 yields

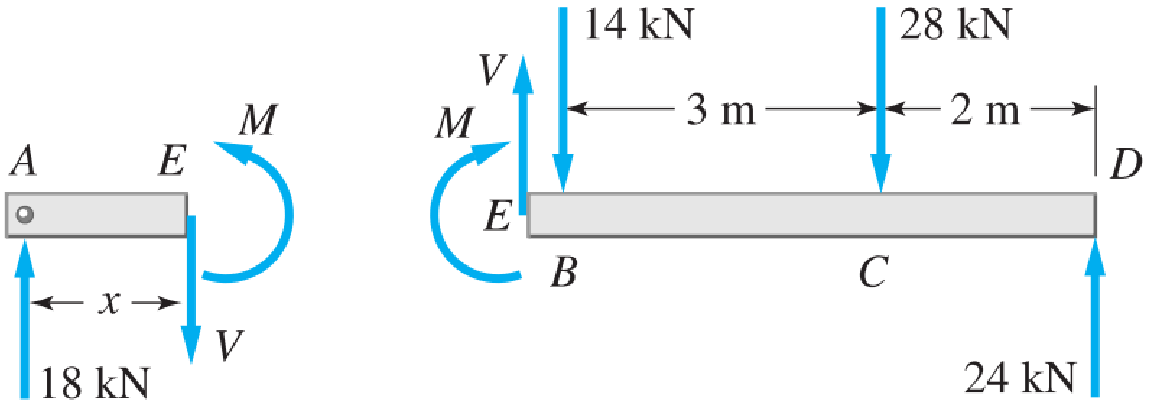


$$\Sigma F_y = 0 \quad +\uparrow \quad 18 - V = 0$$

$$V = +18 \text{ kN}$$

$$\Sigma M_E = 0 \quad +\circlearrowleft \quad -18x + M = 0$$

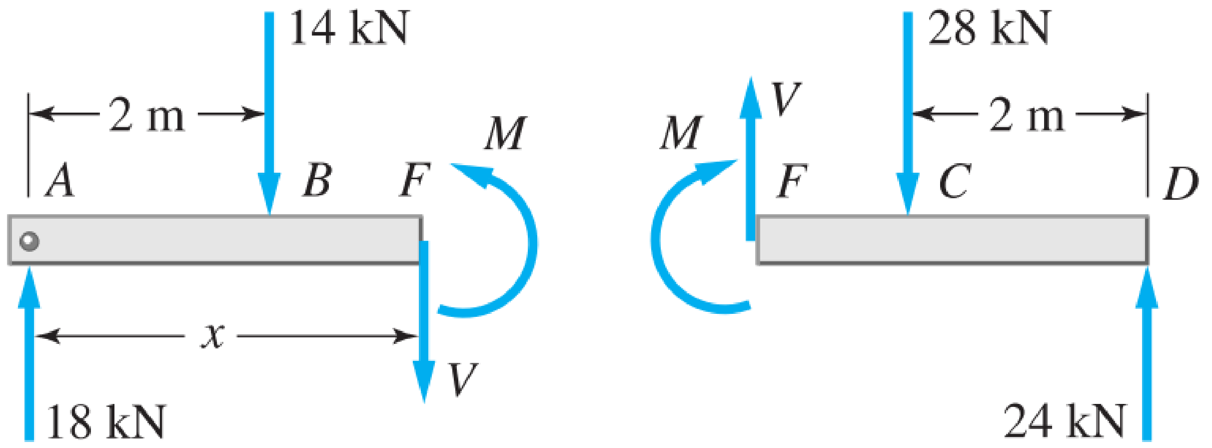
$$M = +18x \text{ kN} \cdot \text{m}$$



(b) FBDs

Segment BC ($2m < x < 5m$)

Figure (c) shows the FBDs for the two parts of the beam that are separated by section 2 , an arbitrary section within segment BC. Once again, V and M are assumed to be positive according to the sign conventions. The analysis of the part to the left of section 2 gives



(c) FBDs

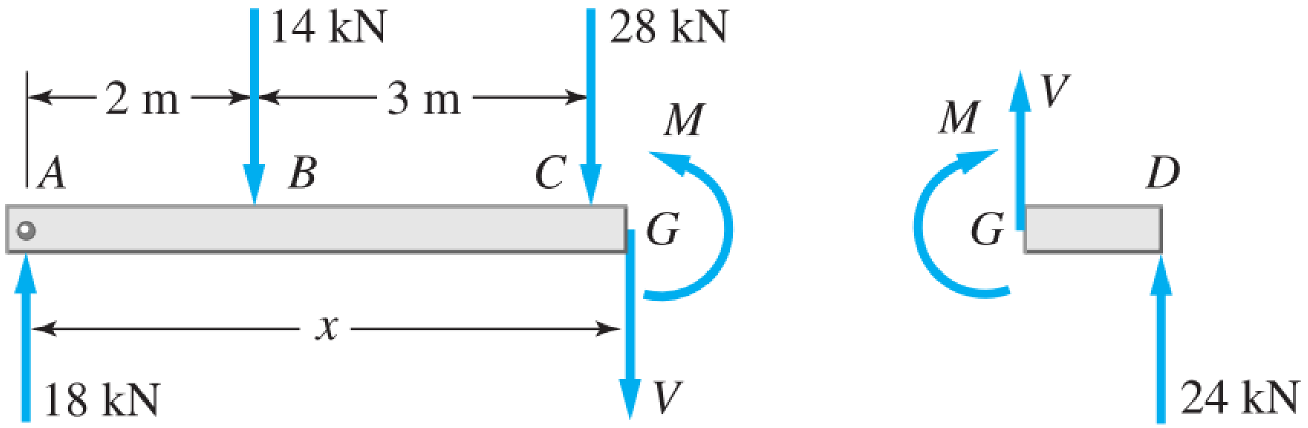
$$\Sigma F_y = 0 \quad +\uparrow \quad 18 - 14 - V = 0$$

$$V = +18 - 14 = +4 \text{ kN}$$

$$\Sigma M_F = 0 \quad +\circlearrowleft \quad -18x + 14(x - 2) + M = 0$$

$$M = +18x - 14(x - 2) = 4x + 28 \text{ kN} \cdot \text{m}$$

Segment CD ($5m < x < 7m$) Section 3 is used to find the shear force and bending moment in segment CD. The FBDs in Fig. (d) again show V and M acting in their positive directions. Analyzing the portion of the beam to the left of section 3 , we obtain



(d) FBDs

$$\Sigma F_y = 0 \quad +\uparrow \quad 18 - 14 - 28 - V = 0$$

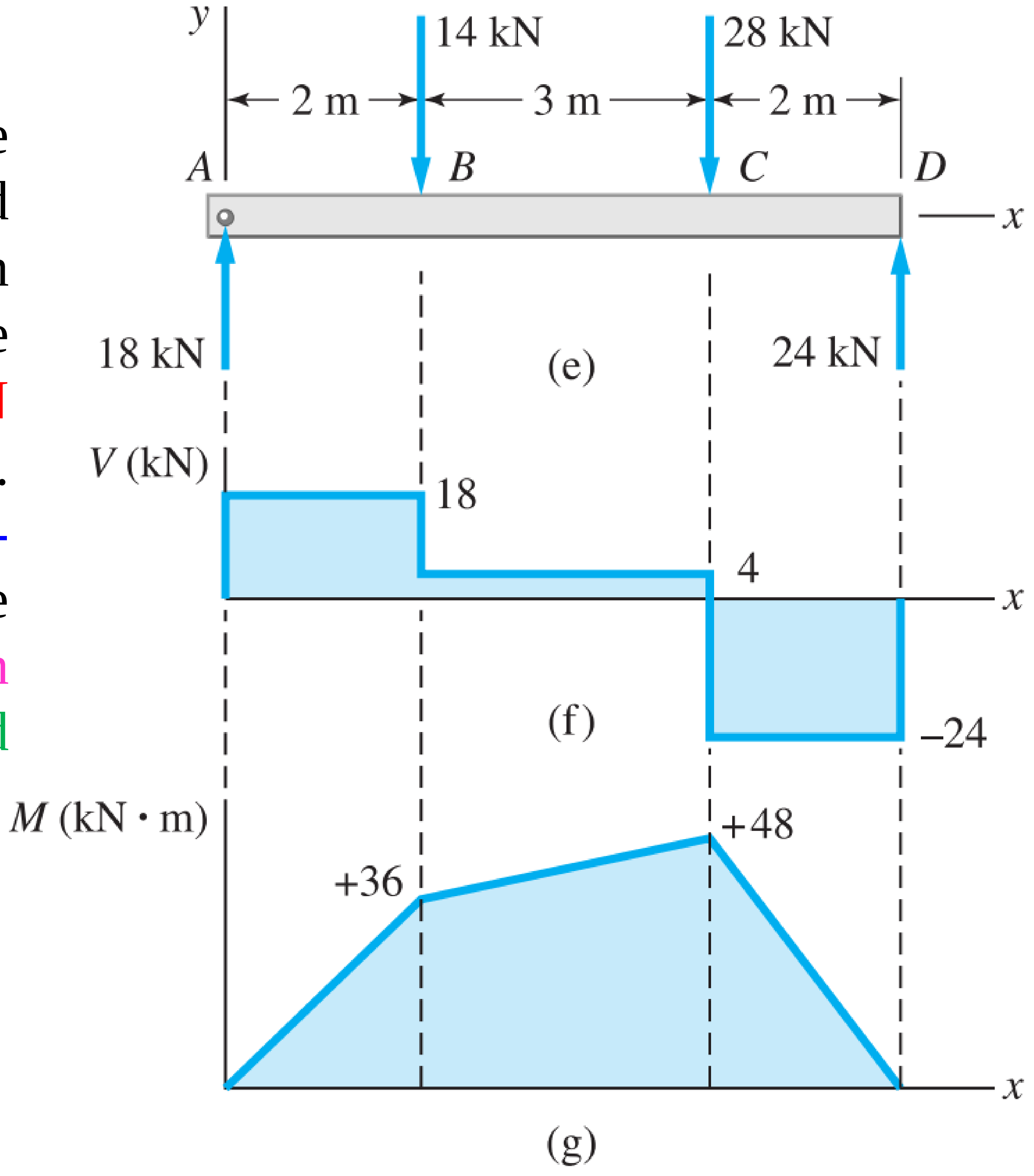
$$V = +18 - 14 - 28 = -24 \text{ kN}$$

$$\Sigma M_G = 0 \quad +\circlearrowleft \quad -18x + 14(x-2) + 28(x-5) + M = 0$$

$$M = +18x - 14(x-2) - 28(x-5) = -24x + 168 \text{ kN} \cdot \text{m}$$

The shear force and bending moment diagrams in Figs. (f) and (g) are the plots of the expressions for V and M derived in Part 1. By placing these plots directly below the sketch of the beam in Fig. (e), we establish a clear visual relationship between the diagrams and locations on the beam.

An inspection of the V-diagram reveals that the largest shear force in the beam is -24 kN and that it occurs at every cross section of the beam in segment CD. From the M-diagram we see that the maximum bending moment is +48 kN m, which occurs under the 28-kN load at C. Note that at each concentrated force the V-diagram "jumps" by an amount equal to the force. Furthermore, there is a discontinuity in the slope of the M-diagram at each concentrated force.



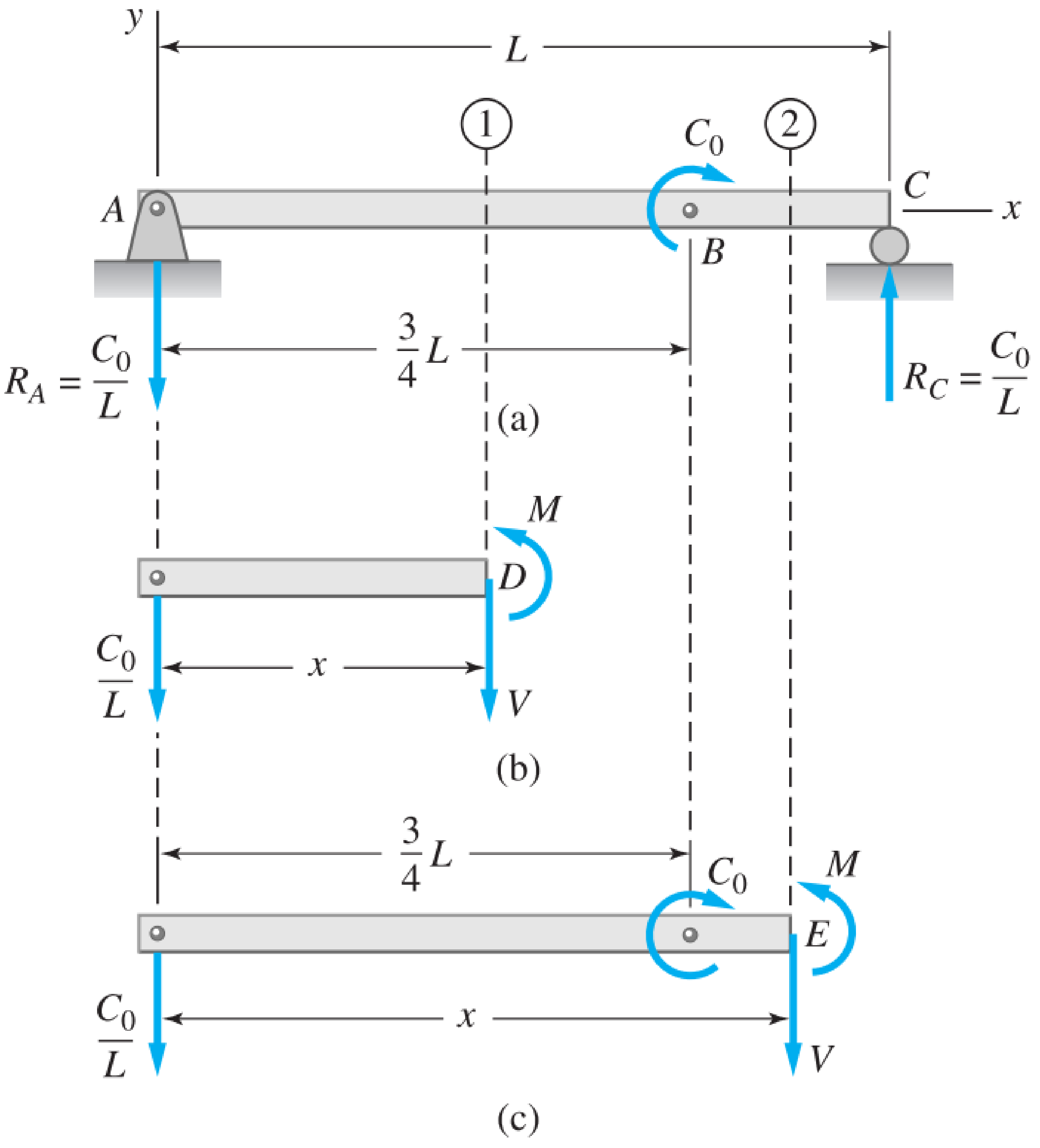
Shear force and bending moment diagrams

Case-II: The simply supported beam in Fig. (a) is loaded by the clockwise couple C_0 at B. (1) Derive the shear force and bending moment equations, and (2) draw the shear force and bending moment diagrams. Neglect the weight of the beam. The support reactions A and C have been computed, and their values are shown in Fig. (a).

Segment AB $(0 < x < \frac{3L}{4})$ Figure (b) shows the FBD of the part of the beam to the left of section 1 (we could also use the part to the right). Note that V and M are assumed to act in their positive directions according to the sign conventions. The equilibrium equations for this portion of the beam yield

$$\Sigma F_y = 0 \quad +\uparrow \quad -\frac{C_0}{L} - V = 0 \qquad V = -\frac{C_0}{L}$$

$$\Sigma M_D = 0 \quad +\circlearrowleft \quad \frac{C_0}{L}x + M = 0 \qquad M = -\frac{C_0}{L}x$$



FBDs

- **Segment BC ($\frac{3L}{4} < x <$ L)** Figure (c) shows the FBD of the portion of the beam to the left of section z2 (the right portion could also be used). Once again, V and M are assumed to act in their positive directions. Applying the equilibrium equations to the beam segment, we obtain

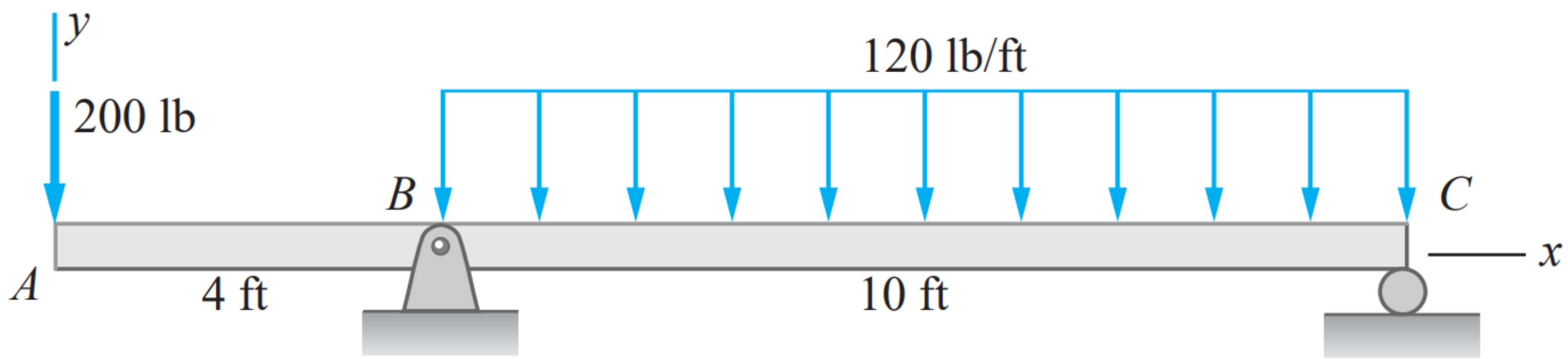
$$\Sigma F_y = 0 \quad +\uparrow \quad -\frac{C_0}{L} - V = 0 \qquad V = -\frac{C_0}{L}$$

$$\Sigma M_E = 0 \quad +\circlearrowleft \quad \frac{C_0}{L}x - C_0 + M = 0 \qquad M = -\frac{C_0}{L}x + C_0$$

- The sketch of the beam is repeated in Fig. (d). The shear force and bending moment diagrams shown in Figs. (e) and (f) are obtained by plotting the expressions for V and M found in Part 1. From the V-diagram, we see that the shear force is the same for all cross sections of the beam. The M-diagram shows a jump of magnitude C_0 at the point of application of the couple.

Case-III: The overhanging beam ABC in Figure carries a concentrated load and a uniformly distributed load. (1) Derive the shear force and bending moment equations; and (2) draw the shear force and bending moment diagrams. Neglect the weight of the beam.

The FBD of the beam is shown in Fig. (b). Note that the uniformly distributed load has been replaced by its resultant, which is the force $120 \times 10 = 1200\ lb$



(area under the loading diagram) acting at the centroid of the loading diagram. The reactions shown at the supports at B and C were computed from the equilibrium equations.

Segment AB ($0 < x < 4\ ft$)

Figure (c) shows the FBD of the portion of the beam that lies to the left of section 1 . (The part of the beam lying to the right of the section could also be used.) The shearing force V and the bending moment M that act at the cut section were assumed to act in their positive directions following the sign conventions. The equilibrium equations for this part of the beam yield

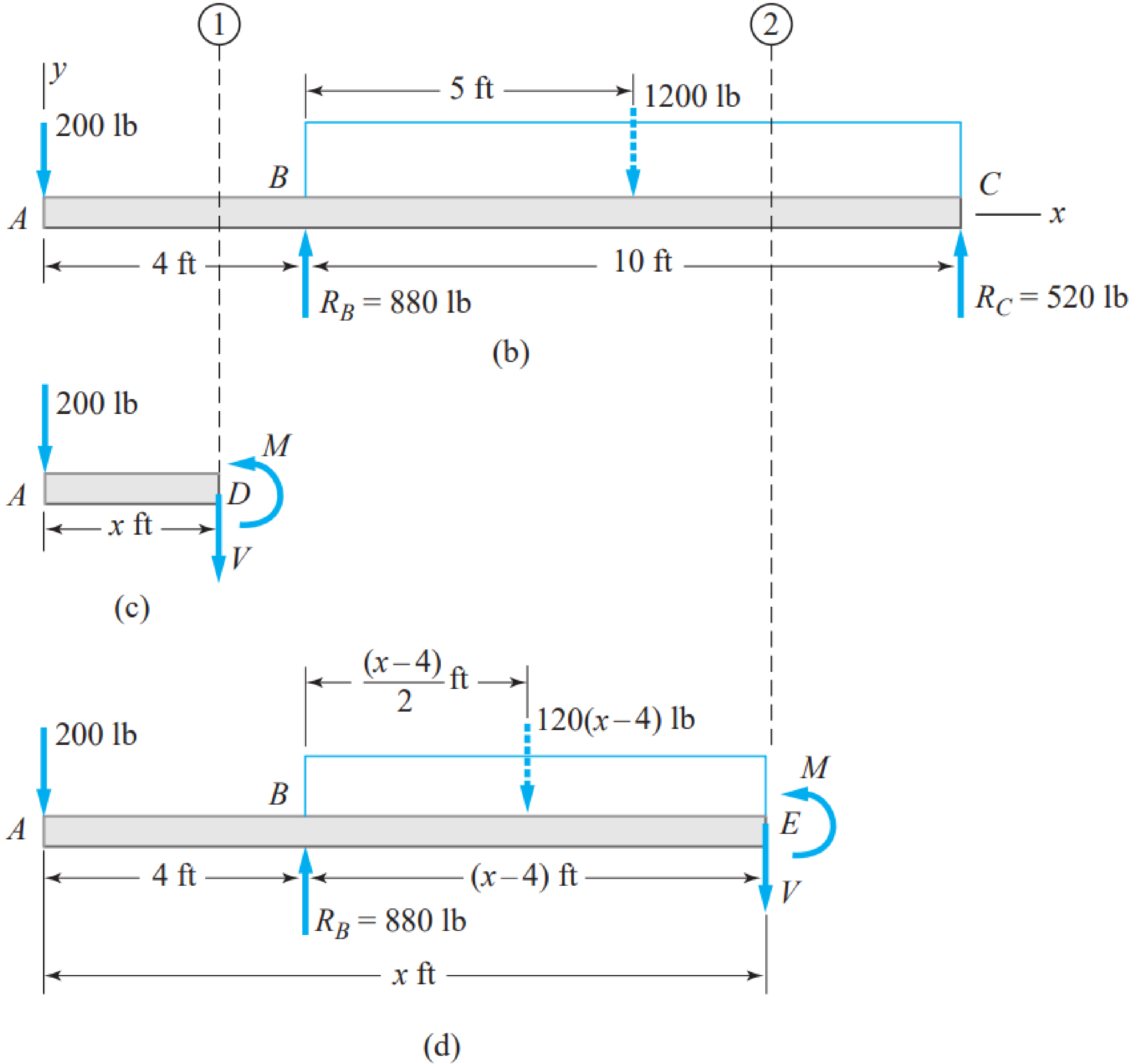
$$\Sigma F_y = 0 \quad +\uparrow \quad -200 - V = 0$$

$$\Sigma M_D = 0 \quad +\circlearrowleft \quad -200x + M = 0$$

$$V = -200 \text{ lb}$$

$$M = -200x \text{ lb} \cdot \text{ft}$$

Segment BC ($4\,ft < x < 14\,ft$) The FBD of the part of the beam that lies to the left of section 2 is shown in Fig. (d). (The portion of the beam lying to the right of the section could also be used.) Once again, the shearing force V and the bending moment M are shown acting in their positive directions. Applying the equilibrium equations to the beam segment, we obtain



$$\Sigma F_y = 0 + \uparrow$$

$$-200 + 880 - 120(x-4) - V = 0$$

$$V = 1160 - 120x \text{ lb}$$

$$\Sigma M_E = 0 + \circlearrowleft$$

$$200x - 880(x-4) + 120(x-4)\frac{(x-4)}{2} + M = 0$$

$$M = -60x^2 + 1160x - 4480 \text{ lb}\cdot\text{ft}$$

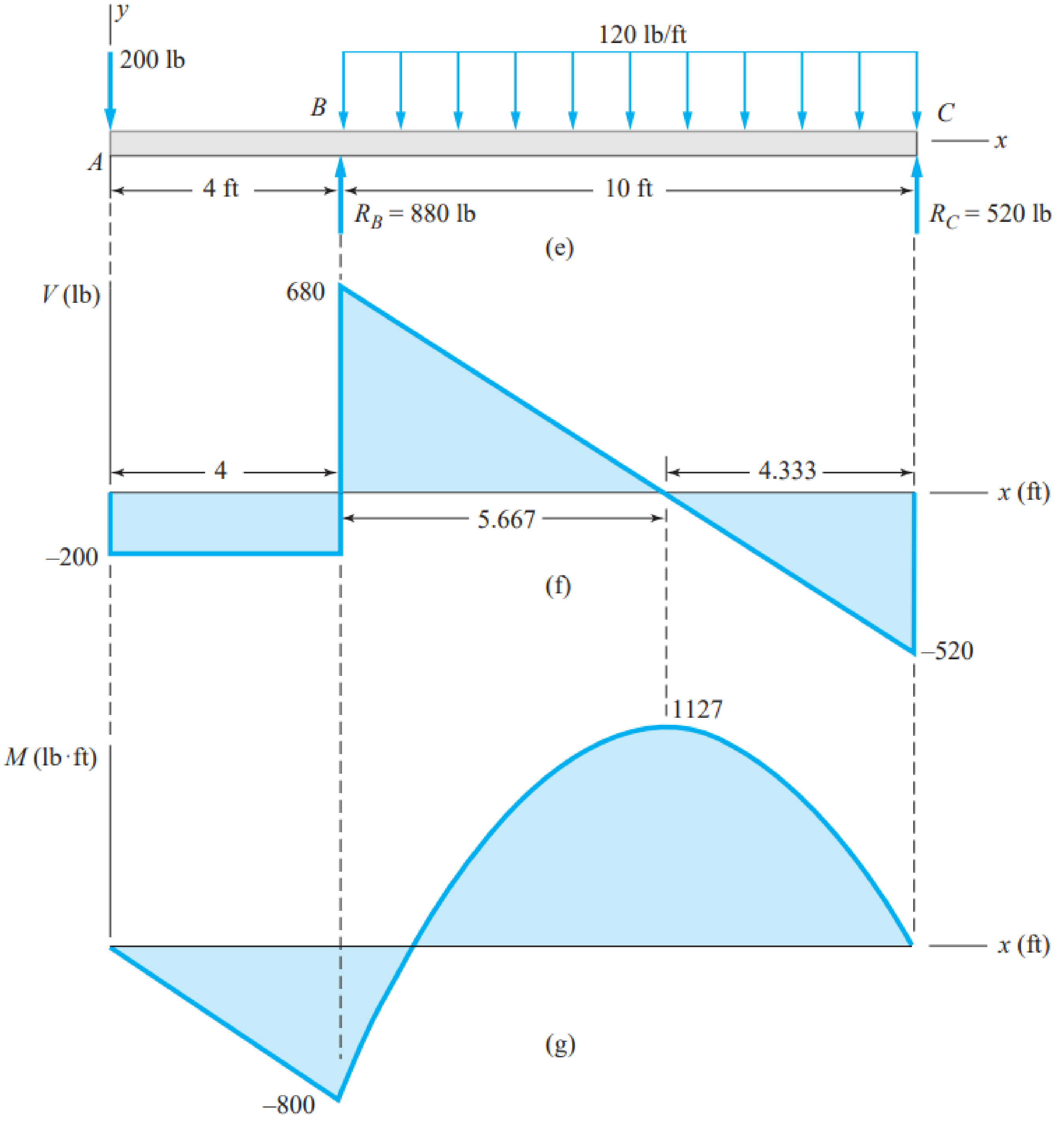
The location of the section where the shear force is zero is determined as follows:

$$V = 1160 - 120x = 0$$

$$x = 9.667 \text{ ft}$$

The maximum bending moment occurs where the slope of the moment diagram is zero; that is, where $\frac{dM}{dx} = 0$, which yields

$$\frac{dM}{dx} = -120x + 1160 = 0$$



- which again gives $x = 9.667\ ft$ (The reason that the maximum bending moment occurs at the section where the shear force is zero) Substituting this value of x into the expression for the bending moment, we find that the maximum bending moment is

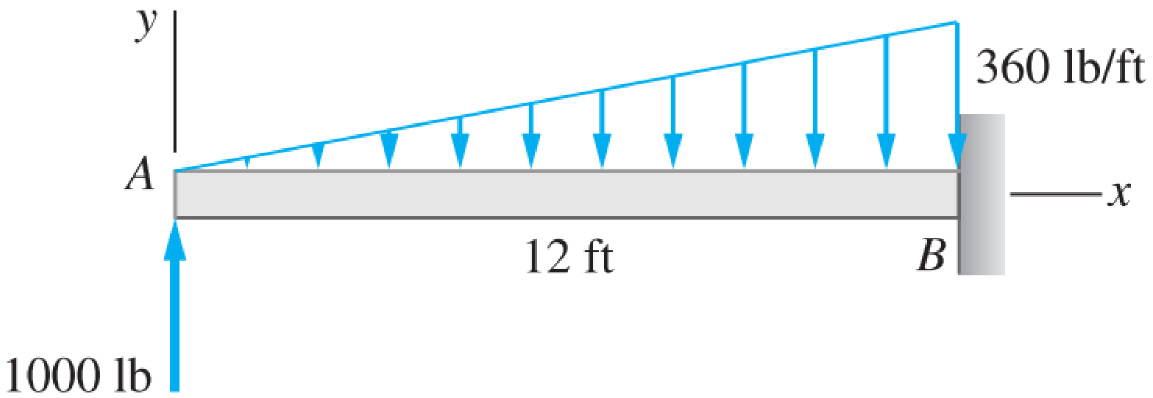
$$M_{\max} = -60(9.667)^2 + 1160(9.667) - 4480 = 1127 \text{ lb} \cdot \text{ft}$$

Case-IV: The cantilever beam in Fig. (a) carries a triangular load, the intensity of which varies from zero at the left end to 360 *lb/ft* at the right end. In addition, a 1000-*lb* upward vertical load acts at the free end of the beam. (1) Derive the shear force and bending moment equations, and (2) draw the shear force and bending moment diagrams. Neglect the weight of the beam.

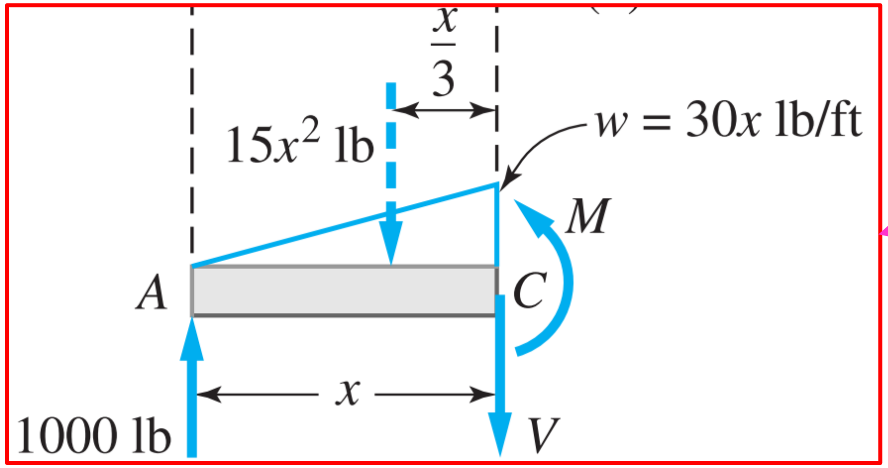
The FBD of the beam is shown in Fig. (b). Note that the triangular load has been replaced by its resultant, which is the force

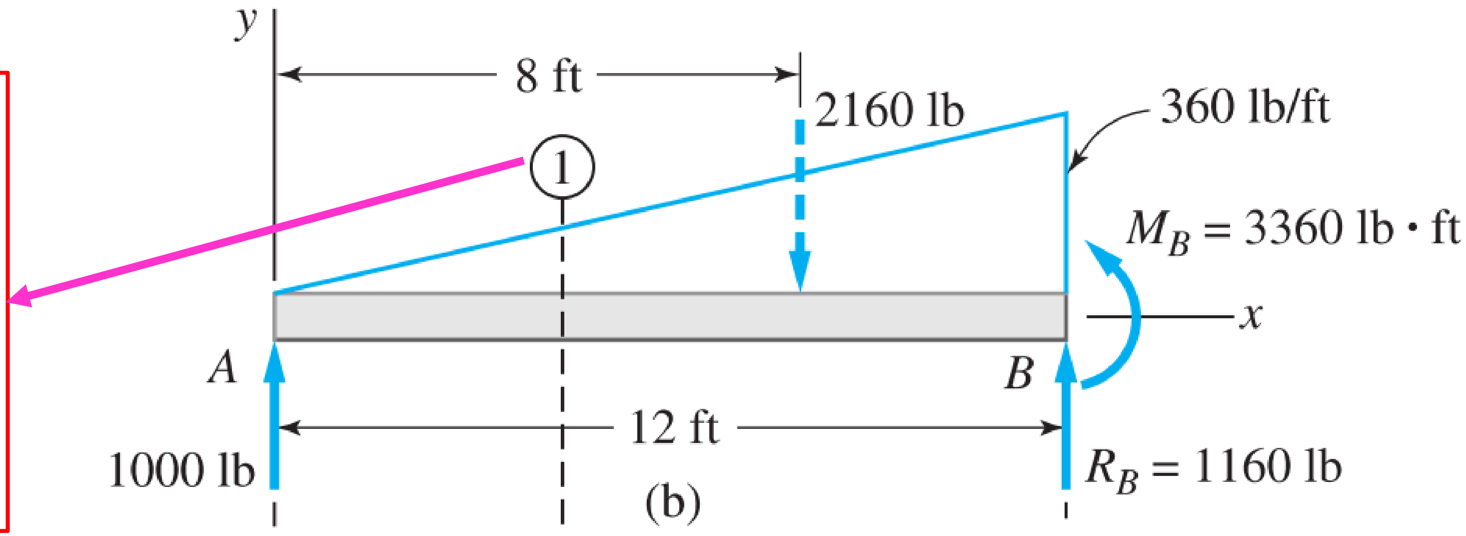
$$\frac{1}{2} \times 12 \times 360 = 2160\ lb$$

(area under the loading diagram) acting at the centroid of the loading diagram.



(a)





(b)

- Because the loading is continuous, the beam does not have to be divided into segments. Therefore, only one expression for V and one expression for M apply to the entire beam.
- Figure (c) shows the FBD of the part of the beam that lies to the left of section 1 . Letting w be the intensity of the loading at section 1 , as shown in Fig. (b), we have from similar triangles,

$$\frac{w}{x} = \frac{360}{12} \quad \text{or} \quad w = 30x\ lb/ft$$

- Now the triangular load in Fig. (c) can be replaced by its resultant force $15x^2\ lb$ acting at the centroid of the loading diagram, which is located at $x = 3\ ft$ from section 1 . The shear force V and bending moment M acting at section 1 are shown acting in their positive directions according to the sign conventions. Equilibrium analysis of the FBD in Fig. (c) yields

$$\Sigma F_y = 0 \quad +\uparrow \quad 1000 - 15x^2 - V = 0$$

$$V = 1000 - 15x^2 \text{ lb}$$

$$\Sigma M_C = 0 \quad +\circlearrowleft \quad -1000x + 15x^2\left(\frac{x}{3}\right) + M = 0$$

$$M = 1000x - 5x^3 \text{ lb} \cdot \text{ft}$$

- Plotting the expressions for V and M found in Part 1 gives the shear force and bending moment diagrams shown in Figs. (b) and (c).
- Observe that the shear force diagram is a parabola and the bending moment diagram is a third-degree polynomial in x. The location of the section where the shear force is zero is found from

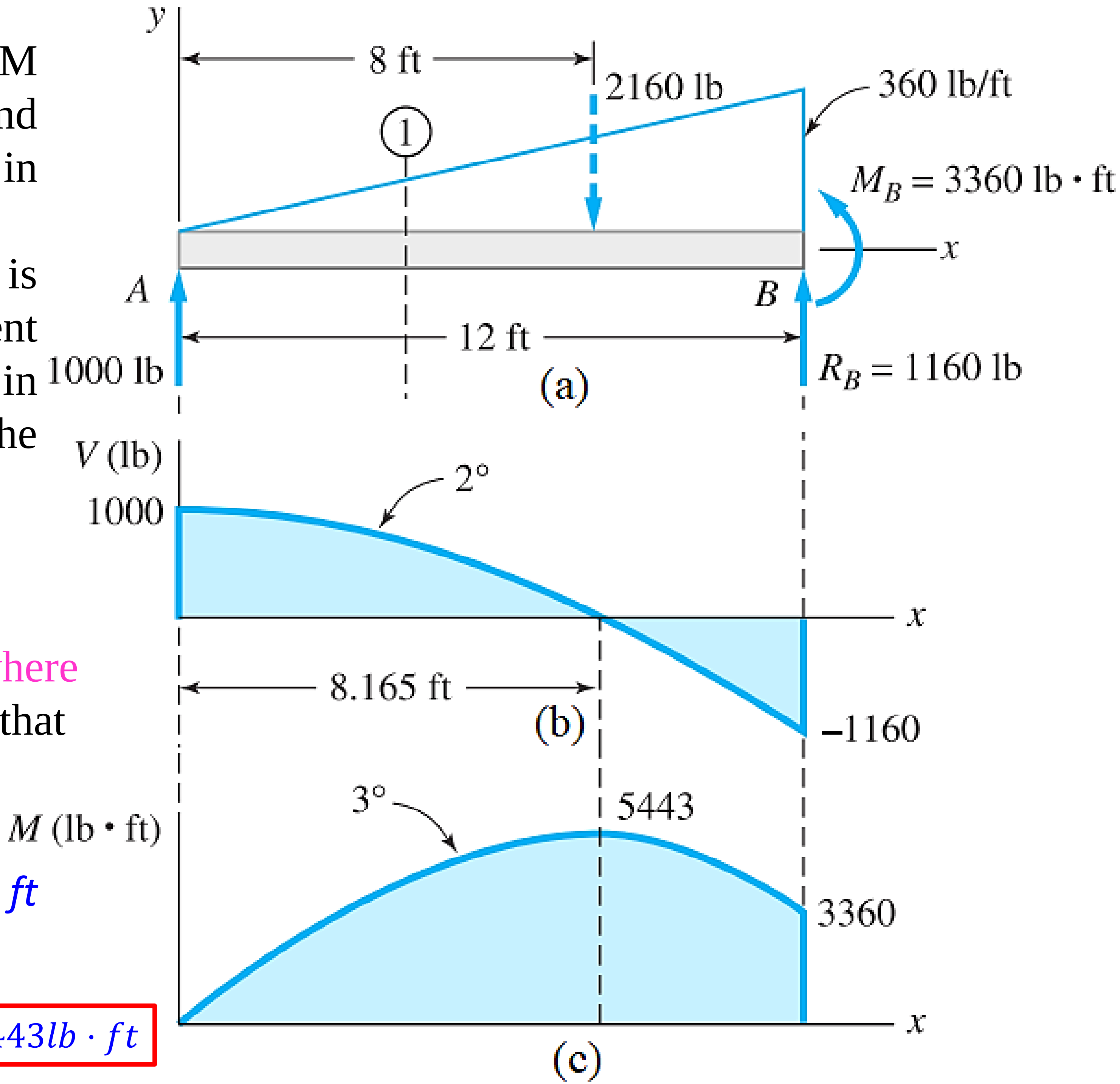
$$V = 1000 - 15x^2 = 0$$

which gives $x = 8.165\ ft$

The maximum bending moment occurs where the slope of the moment diagram is zero; that is, where $\frac{dM}{dx} = 0$, which yields

$$\frac{dM}{dx} = 1000 - 15x^2 = 0 \qquad x = 8.165\ ft$$

Hence $M_{max} = 1000(8.165) - 5(8.165)^3 = 5443 lb \cdot ft$



Bending stresses in beam

Flexure formula

or

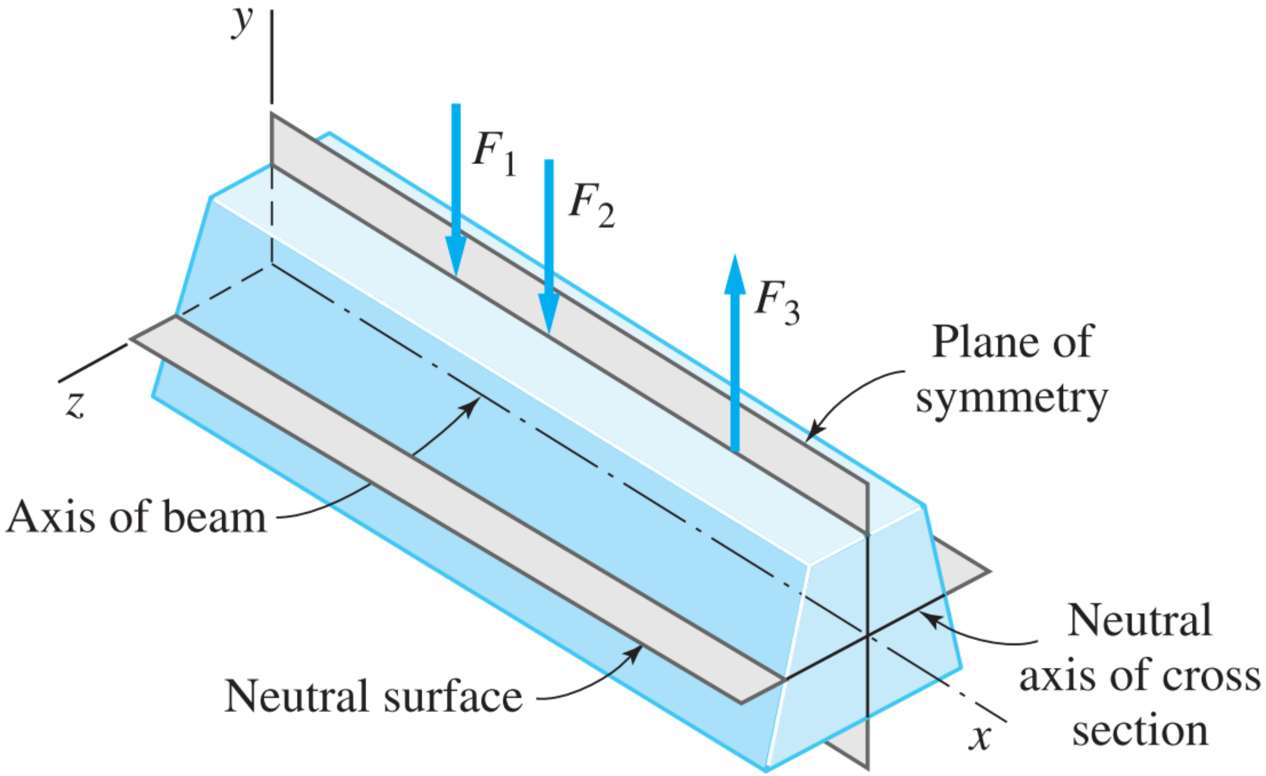
Bending equation

- ❖ In previous chapters, we considered stresses in bars caused by axial loading and torsion. Here we introduce the third fundamental loading: bending.
- ❖ When deriving the relationships between the bending moment and the stresses it causes, we find it again necessary to make certain simplifying assumptions.
- ❖ Although these assumptions may appear to be overly restrictive, the resulting equations have served well in the design of straight, elastic beams.
- ❖ Here We use the same steps in the analysis of bending that we used for torsion
  - Make simplifying assumptions about the deformation based upon experimental evidence.
  - Determine the strains that are geometrically compatible with the assumed deformations.
  - Use Hooke's law to express the equations of compatibility in terms of stresses.
  - Derive the equations of equilibrium. (These equations provide the relationships between the stresses and the applied loads.)

Simplifying Assumptions

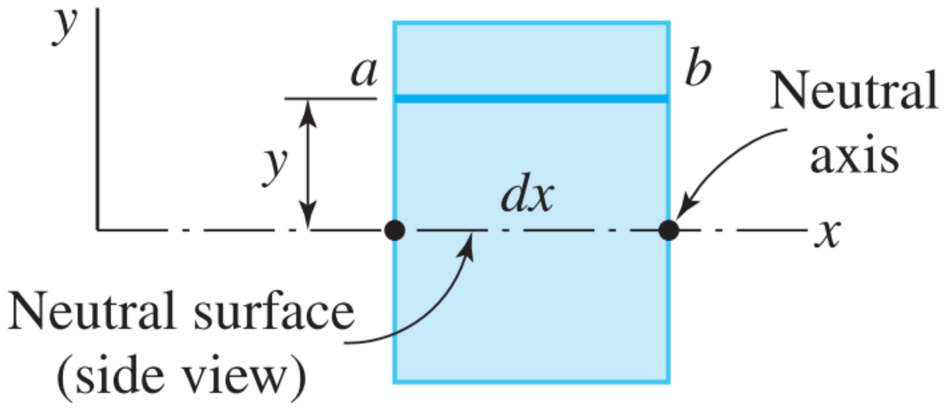
The stresses caused by the bending moment are known as bending stresses, or flexure stresses. The relationship between these stresses and the bending moment is called the flexure formula. In deriving the flexure formula, we make the following assumptions:

- The beam has an axial plane of symmetry, which we take to be the xy-plane shown in Fig.
- The applied loads (such as F1, F2, and F3 in Fig.) lie in the plane of symmetry and are perpendicular to the axis of the beam (the x-axis).
- The axis of the beam bends but does not stretch (the axis lies somewhere in the plane of symmetry; its location will be determined later).
- Plane sections of the beam remain plane (do not warp) and perpendicular to the deformed axis of the beam.
- Material of the beam is homogenous and isotropic.
- The beam is straight before the bending (bending equation derived based on this assumption is valid on straight beams only)



- ❖ Because the shear stresses caused by the vertical shear force will distort (warp) an originally plane section, we are limiting our discussion here to the deformations caused by the bending moment alone.
- ❖ However, it can be shown that the deformations due to the vertical shear force are negligible in slender beams (the length of the beam is much greater than the cross sectional dimensions) compared to the deformations caused by bending.
- ❖ The above assumptions lead us to the following conclusion:
    - Each cross section of the beam rotates as a rigid entity about a line called the neutral axis of the cross section.
    - The neutral axis passes through the axis of the beam and is perpendicular to the plane of symmetry, as shown in Fig.
    - The xz-plane that contains the neutral axes of all the cross sections is known as the neutral surface of the beam

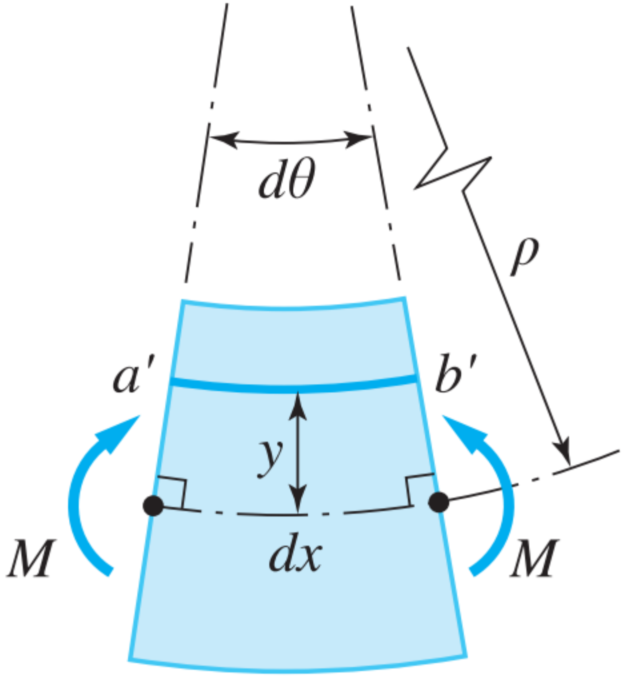
Compatibility condition



Before deformation

Figure shows a segment of the beam bounded by two cross sections that are separated by the infinitesimal distance dx. Due to the bending moment M caused by the applied loading, the cross sections rotate relative to each other by the amount dθ.

The radius of curvature of the deformed surface is denoted by ρ. Note that the distance between the cross sections, measured along the neutral surface, remains unchanged at dx (it is assumed that the axis of the beam does not change length). Therefore, the longitudinal fibers lying on the neutral surface are undeformed, whereas the fibers above the surface are compressed and the fibers below are stretched.



After deformation

Let the undeformed length of fiber lies a distance y above the neutral surface is $ab = \mathrm{d}x$. In the deformed state, the fiber forms the arc $a'b'$ of radius $(\rho - y)$, subtended by the angle dθ. Therefore, its deformed length is

$$a'b' = (\rho - y)\,\mathrm{d}\theta$$

The original length of this fiber is $ab = \mathrm{d}x = \rho\,\mathrm{d}\theta$. The normal strain of the fiber is found by dividing the change in length by the original length, yielding

$$\epsilon = \frac{a'b' - ab}{ab} = \frac{(\rho - y)\,\mathrm{d}\theta - \rho\,\mathrm{d}\theta}{\rho d\theta} = -\frac{y}{\rho}$$

Assuming that the stress is less than the proportional limit of the material, we can obtain the normal stress in fiber ab from Hooke's law:

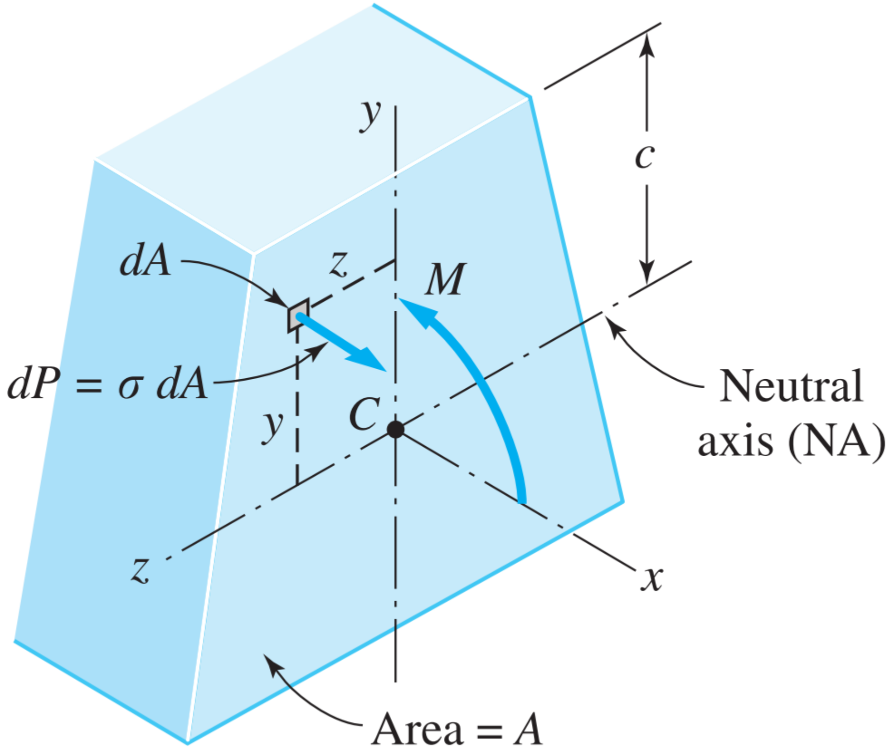
$$\sigma = \epsilon E = -\frac{E}{\rho} y$$

The normal stress of a longitudinal fiber is proportional to the distance y of the fiber from the neutral surface. The negative sign indicates that positive bending moment causes compressive stress when y is positive (fibers above the neutral surface) and tensile stress when y is negative (fibers below the neutral surface), as expected.

Equilibrium condition

To complete the derivation of the flexure formula, we must locate the neutral axis of the cross section and derive the relationship between r and M.

Both tasks can be accomplished by applying the equilibrium conditions. Figure shows a typical cross section of a beam. The normal force acting on the infinitesimal area dA of the cross section is



$$dP = \sigma\, dA.$$

$$dP = -\frac{E}{\rho} y\, dA$$

where y is the distance of dA from the neutral axis (NA). Equilibrium requires that the resultant of the normal stress distribution over the cross section must be equal to the bending moment M acting about the neutral axis (z-axis). In other words,

$$-\int_A y\, dP = M \qquad \ldots\ldots (i)$$

where the integral is taken over the entire cross-sectional area *A* (the minus sign in the expression is needed because the moment of *dP* and positive *M* have opposite sense).

Moreover, the resultant axial force and the resultant bending moment about the y-axis must be zero; i.e.,

$$\int_A \mathrm{d}P = 0 \qquad \dots\dots..(ii)$$

$$and$$

$$\int_A z\,\mathrm{d}P = 0 \qquad \dots\dots..(\mathrm{iii})$$

These three equilibrium equations are developed in detail below

Resultant Axial Force Must Vanish (The condition for zero axial force is)

$$\int_A \mathrm{d}P = -\frac{E}{\rho}\int_A y\,\mathrm{d}A = 0$$

Because $\frac{E}{\rho} \neq 0$, this equation can be satisfied only if

$$\int_A y\,\mathrm{d}A = 0$$

The integral in equation is the first moment of the cross-sectional area about the neutral axis.

"It can be zero only if the neutral axis passes through the centroid C of the cross-sectional area".

Hence, the condition of zero axial force locates the neutral axis of the cross section.

Resultant Moment About y-Axis Must Vanish: This condition is

$$\int_A z\,\mathrm{d}P = -\frac{E}{\rho}\int_A zy\,\mathrm{d}A = 0$$

The integral $\int_A zy\,\mathrm{d}A$ is the product of inertia of the cross-sectional area. According to our assumptions, the y-axis is an axis of symmetry for the cross section, in that case this integral is zero and equation is automatically satisfied.

Resultant Moment About the Neutral Axis Must Equal M: Equating the resultant moment about the z-axis to M gives us

$$-\int_A y\,\mathrm{d}P = \frac{E}{\rho}\int_A y^2\,\mathrm{d}A = M$$

Recognizing that $\int_A y^2\,\mathrm{d}A = I$ is the moment of inertia1 of the cross-sectional area about the neutral axis (the z-axis), we obtain the moment-curvature relationship

$$M = \frac{EI}{\rho} \qquad \text{and} \qquad \frac{1}{\rho} = \frac{M}{EI}$$

Substituting the expression for $\frac{1}{\rho}$, we get the flexure formula:

$$\sigma = -\frac{M}{I}y$$

Note that a positive bending moment M causes negative (compressive) stress above the neutral axis and positive (tensile) stress below the neutral axis, as discussed previously. The maximum value of bending stress without regard to its sign is given by

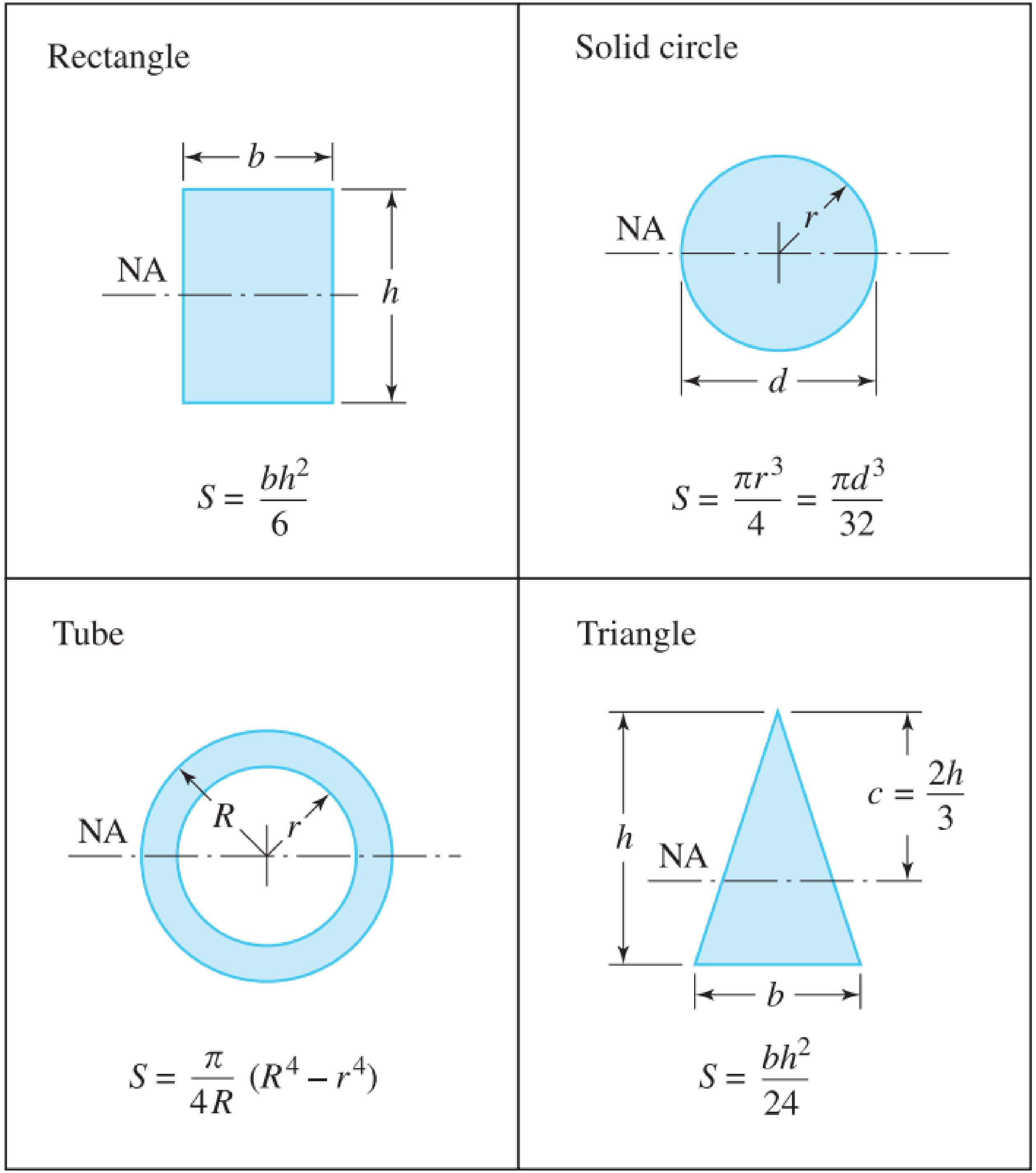
$$\sigma_{\max} = -\frac{|M|_{\max}c}{I}$$

where $|M|_{\max}$ is the largest bending moment in the beam regardless of sign, and c is the distance from the neutral axis to the outermost point of the cross section.

$$\sigma_{\max} = -\frac{|M|_{\max}}{S}$$

where $S = I/c$ is called the section modulus of the beam. Its unit is m^3.

Section moduli of common cross sections



Procedure for determining the bending stresses

- Use the method of sections to determine the bending moment M (with its correct sign) at the cross section containing the given point.
- Determine the location of the neutral axis.
- Compute the moment of inertia I of the cross sectional area about the neutral axis.
- Determine the y-coordinate of the given point. Note that y is positive if the point lies above the neutral axis and negative if it lies below the neutral axis.
- Compute the bending stress from $\sigma = -\frac{M}{I}y$. If correct signs are used for M and y, the stress will also have the correct sign (tension positive, compression negative).

Maximum Bending Stress: Symmetric Cross Section

If the neutral axis is an axis of symmetry of the cross section, the maximum tensile and compressive bending stresses in the beam are equal in magnitude and occur at the section of the largest bending moment. The following procedure is recommended for determining the maximum bending stress in a prismatic beam:

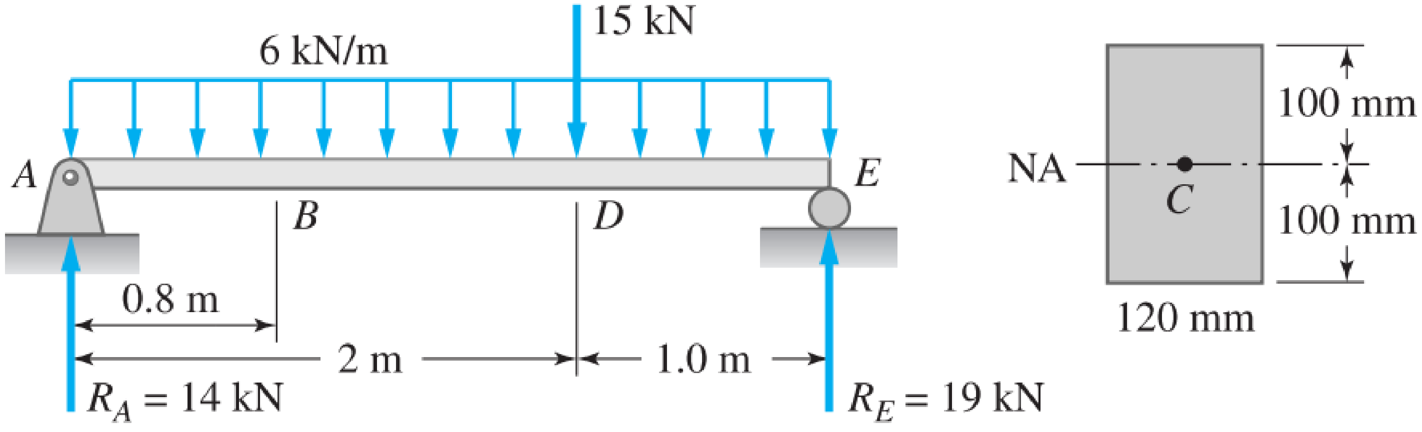
- Draw the bending moment diagram. Identify the bending moment $|M|_{\max}$ that has the largest magnitude.
- Compute the moment of inertia I of the cross-sectional area about the neutral axis.
- Calculate the maximum bending stress from $\sigma_{\max} = -\frac{|M|_{\max} c}{I} = -\frac{|M|_{\max}}{S}$, where c is the distance from the neutral axis to the top or bottom of the cross section.

Maximum Tensile and Compressive Bending Stresses: Unsymmetrical Cross Section

If the neutral axis is not an axis of symmetry of the cross section, the maximum tensile and compressive bending stresses may occur at different sections. The recommended procedure for computing these stresses in a prismatic beam follows:

- Draw the bending moment diagram. Identify the largest positive and negative bending moments.
- Determine the location of the neutral axis and record the distances c_{top} and c_{bot} from the neutral axis to the top and bottom of the cross section.
- Compute the moment of inertia I of the cross section about the neutral axis.
- Calculate the bending stresses at the top and bottom of the cross section where the largest positive bending moment occurs from $\sigma = -\frac{M}{I}y$. At the top of the cross section, where $y = c_{top}$, we obtain $\sigma_{top} = -\frac{Mc_{top}}{I}$. At the bottom of the cross section, we have $y = c_{bot}$, so that $\sigma_{bot} = \frac{Mc_{bot}}{I}$. Repeat the calculations for the cross section that carries the largest negative bending moment. Inspect the four stresses thus computed to determine the largest tensile (positive) and compressive (negative) bending stresses in the beam.

Que. 1. The simply supported beam in Fig. (a) has a rectangular cross section 120 mm wide and 200 mm high.
(1) Compute the maximum bending stress in the beam.
(2) Sketch the bending stress distribution over the cross section on which the maximum bending stress occurs.
(3) Compute the bending stress at a point on section B that is 25 mm below the top of the beam.



Before we can find the maximum bending stress in the beam, we must find the maximum bending moment. We begin by computing the external reactions at A and E; the results are shown in Fig. (a). Then we sketch the shear force and bending moment diagrams, obtaining the results in Figs. (b) and (c).

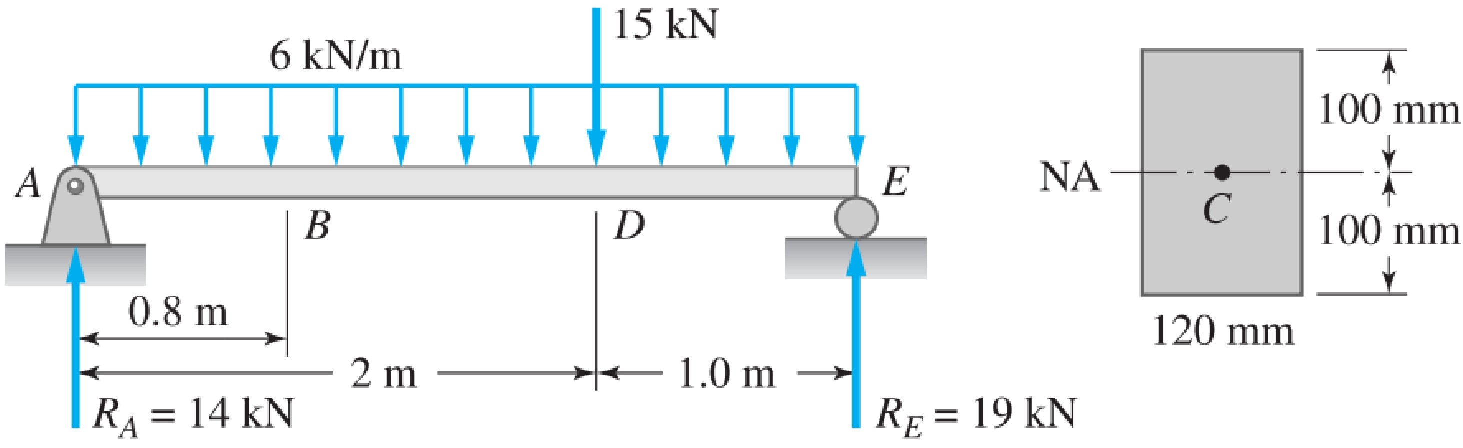
We see that the maximum bending moment is $M_{\max} = 16\ kN \cdot m$, occurring at D. In this case, the neutral axis (NA) is an axis of symmetry of the cross section, as shown in Fig. (a). The moment of inertia of the cross section about the neutral axis is

$$I = \frac{bh^3}{12} = \frac{0.12(0.2)^3}{12} = 80 \times 10^{-6} m^4$$

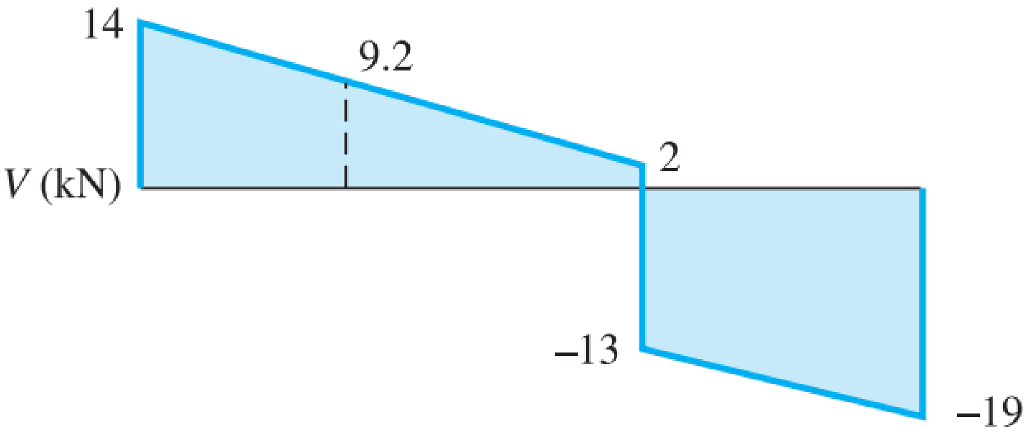
and the distance between the neutral axis and the top (or bottom) of the cross section is c = 100 mm = 0.1 m.

Part 1

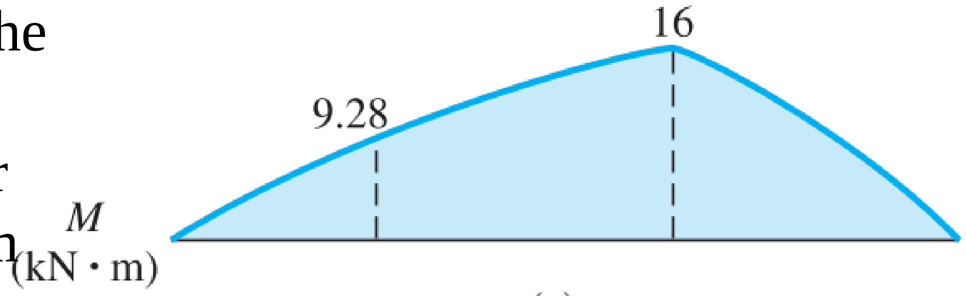
The maximum bending stress in the beam occurs on the cross section that carries the largest bending moment, which is at D. Using the flexure formula, we obtain for the maximum bending stress in the beam



(a)



(b)



(c)

$$\sigma_{\max} = -\frac{|M|_{\max} c}{I} = \frac{16 \times 10^3 (0.1)}{80 \times 10^{-6}} = 20 \times 10^3\ Pa = 20\ MPa$$

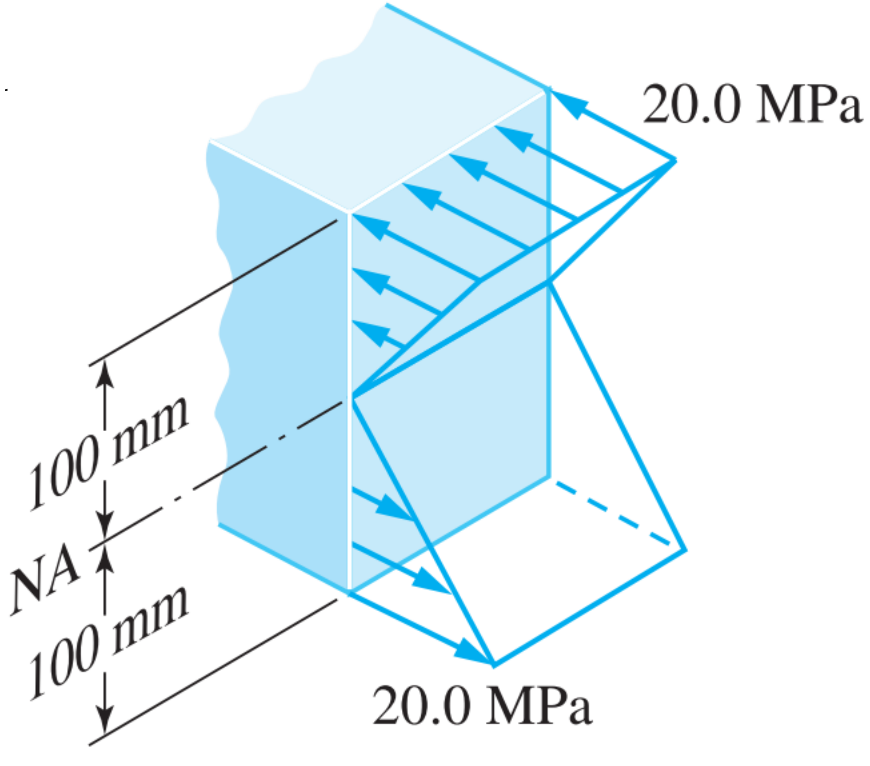
Part 2

The stress distribution on the cross section at D is shown in Fig. (d). When drawing the figure, we were guided by the following observations:

(i) The bending stress varies linearly with distance from the neutral axis.

(ii) Because $M_{\max}$ is positive, the top half of the cross section is in compression and the bottom half is in tension; and

(iii) Due to symmetry of the cross section about the neutral axis, compressive stresses are equal in magnitude.



(d)

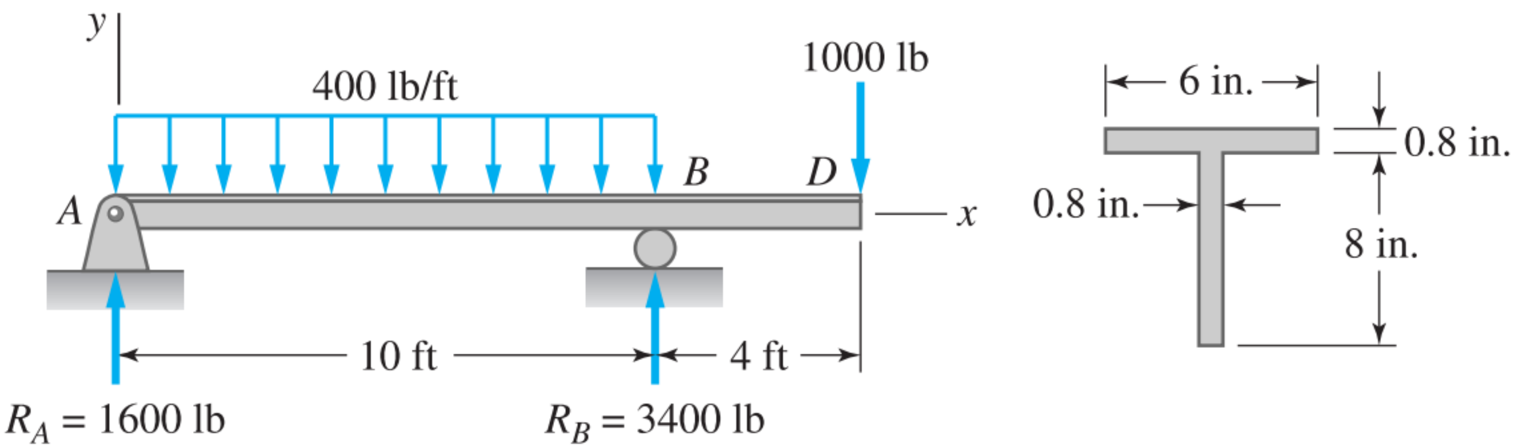
Part 3

From Fig. (c) we see that the bending moment at section B is $M = 9.28\ kN\ \ m$. The y-coordinate of the point that lies 25 mm below the top of the beam is $y = 100 - 25 = 75\ mm = 0.075\ m$. If we substitute these values into above equation, the bending stress at the specified location becomes

$$\sigma = -\frac{M}{I}y = -\frac{9.28 \times 10^3(0.075)}{80 \times 10^{-6}} - 8.7 \times 10^6\ Pa = 8.7\ MPa$$

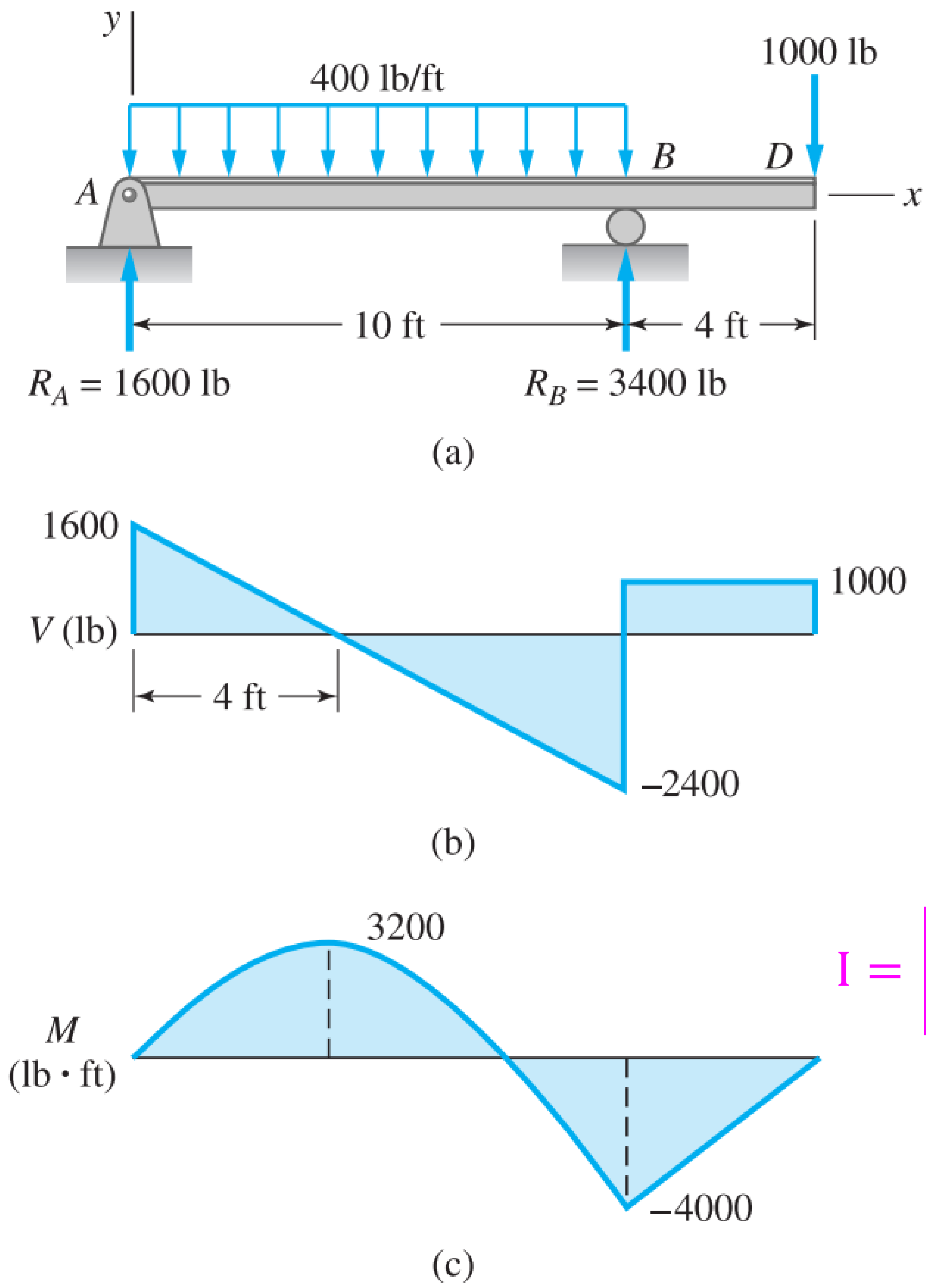
The negative sign indicates that this bending stress is compressive, which is expected because the bending moment is positive and the point of interest lies above the neutral axis.

Que.2. The simply supported beam in Fig. (a) has the T-shaped cross section shown. Determine the values and locations of the maximum tensile and compressive bending stresses.



Preliminary Calculations:

Before we can find the maximum tensile and compressive bending stresses, we must find the largest positive and negative bending moments. Therefore, we start by computing the external reactions at A and B, and then sketch the shear force and bending moment diagrams. The results are shown in Figs. (a)–(c). From Fig. (c), we see that the largest positive and negative bending moments are 3200 lb-ft and 4000 lb-ft, respectively. Because the cross section does not have a horizontal axis of symmetry, we must next locate the neutral (centroidal) axis of the cross section. As shown in Fig. (d), we consider the cross section to be composed of the two rectangles with areas $A_1 = 0.8(8) = 6.4\ in^2$ and $A_2 = 0.8(6) = 4.8\ in^2$.



The centroidal coordinates of the areas are $\overline{y_1} = 4\ in.\ and\ \overline{y_2} = 8.4\ in.$ measured from the bottom of the cross section. The coordinate $\bar{y}$ of the centroid C of the cross section is

$$\bar{y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A_1 + A_2} = \frac{64(4) + 48(8.4)}{6.4 + 48} = 5.886\ \text{in.}$$

We can now compute the moment of inertia I of the cross-sectional area about the neutral axis. Using the parallel-axis theorem, we have

$$I = \sum[\bar{I}_i + A_i(\bar{y}_i - \bar{y})^2],$$

where $\overline{I_i} = \frac{b_i h_i}{12}$ is the moment of inertia of a rectangle about its own centroidal axis. Thus,

$$\text{I} = \left[\frac{0.8(8)^3}{12} + 6.4(4 - 5.886)^2\right] + \left[\frac{6(0.8)^3}{12} + 4.8(8.4 - 5 \cdot 886)^2\right]$$

$$\boxed{\text{I} = 87.49\ in^4}$$

Maximum Bending Stresses

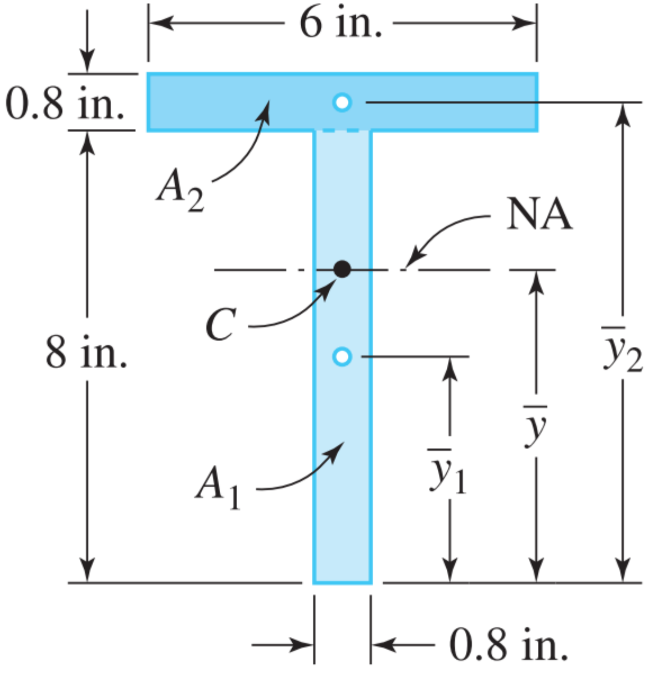
The distances from the neutral axis to the top and the bottom of the cross section are $c_{top} = 8.8 - \bar{y} = 2.914\ in.$ and $c_{bot} = -\bar{y} = -5.886\ in.$, as shown in Fig. (e). Because these distances are different, we must investigate stresses at two locations: at $x = 4\ ft$ (where the largest positive bending moment occurs) and at $x = 10\ ft$ (where the largest negative bending moment occurs).

Stresses at $x = 4\ ft$:

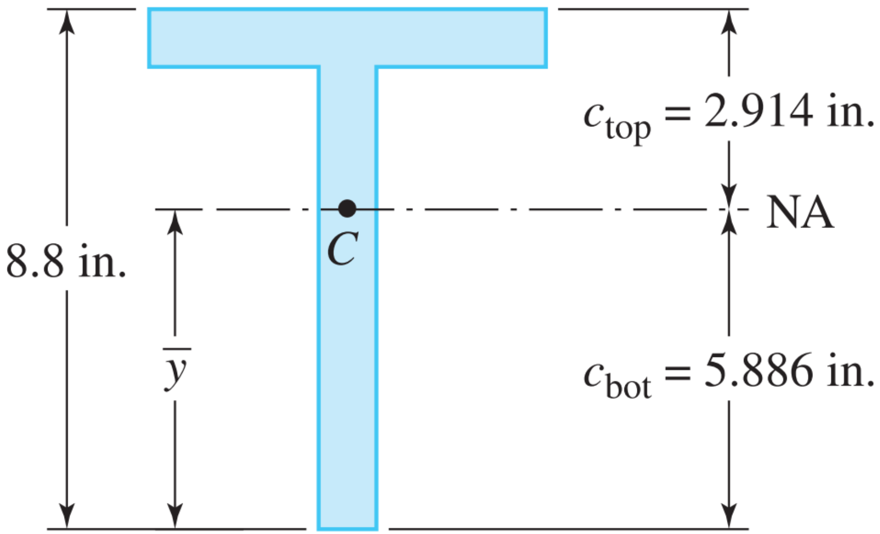
The bending moment at this section is $M = 3200$ *lb-ft*, causing compression above the neutral axis and tension below the axis. The resulting bending stresses at the top and bottom of the cross section are

$$\sigma_{top} = -\frac{Mc_{top}}{I} = -\frac{(3200 \times 12) \times 2.914}{87.49} = -1279\ psi$$

$$\sigma_{bot} = -\frac{Mc_{bot}}{I} = -\frac{(3200 \times 12) \times (-5.886)}{87.49} = 2580\ psi$$



(d)



(e)

Stresses at $x = 10\ ft$:

The bending moment at this section is *M* = - *4000 lb-ft*, resulting in tension above the neutral axis and compression below the neutral axis. The corresponding bending stresses at the extremities of the cross section are

$$\sigma_{top} = -\frac{Mc_{top}}{I} = -\frac{(-4000 \times 12) \times 2.914}{87.49} = 1599\ psi$$

$$\sigma_{bot} = -\frac{Mc_{bot}}{I} = -\frac{(-4000 \times 12) \times (-5.886)}{87.49} = -3230\ psi$$

Inspecting the above results, we conclude that the maximum tensile and compressive stresses in the beam are

$$(\sigma_T)_{\max} = 2580\ psi\ \ (bottom\ of\ the\ section\ at\ x = 4\ ft)$$

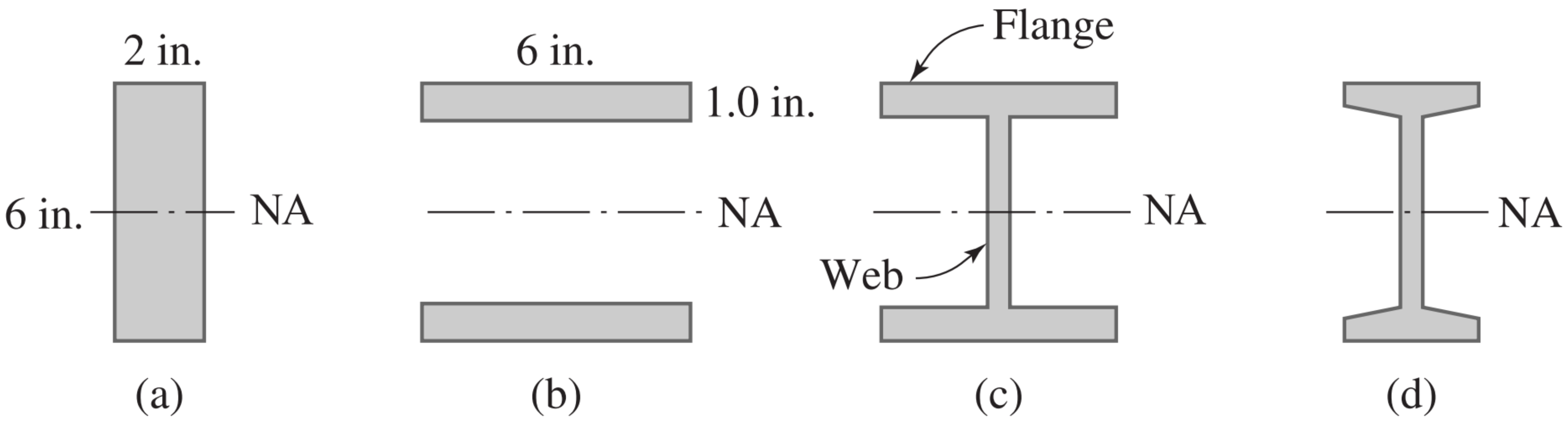
$$(\sigma_C)_{\max} = 3230\ psi\ \ (bottom\ of\ the\ section\ at\ x = 10\ ft)$$

Economic Section

The portions of a beam located near the neutral surface are understressed compared with those at the top or bottom.

Therefore, beams with certain cross-sectional shapes (including a rectangle and a circle) utilize the material inefficiently because much of the cross section contributes little to resisting the bending moment.

Consider, for example, a beam with the rectangular cross section shown in Fig. 5.5(a).



The section modulus of this beam is $S = \frac{bh^2}{6} = \frac{2(6)^2}{6} = 12\ in^3$.

- If the working stress is $\sigma_w = 18\ ksi$, the maximum safe bending moment for the beam is

$$M = \sigma_w S = 18 \times 12 = 216\ kip.in.$$

- In Fig. 5.5(b), we have rearranged the area of the cross section but kept the same overall depth. It can be shown that the section modulus has increased to $S = 25.3\ in^3$. Thus, the new maximum allowable moment is

$$M = \sigma_w S = 18 \times 25.3 = 455\ kip.in.$$

- This moment is more than twice the allowable moment for the rectangular section of the same area. This increase in moment-carrying capacity is caused by more cross-sectional area being located at a greater distance from the neutral axis.
- However, the section in Fig. 5.5(b) is not practical because its two parts, called the flanges, are disconnected and thus would not act as an integral unit. It is necessary to use some of the area to attach the flanges to each other, as in Fig. 5.5(c).
- The vertical connecting piece is known as the web of the beam. Figure 5.5(c) is similar to a wide-flange beam, referred to as a W-shape. A W-shape is one of the most efficient standard structural shapes manufactured because it provides great *flexural strength* with *minimum weight of material*.

- Another “slimmer” version of this shape is the I-beam (referred to as an S-shape) shown in Fig. 5.5(d). The I-beam preceded the wide-flange beam, but because it is not as e‰cient, it has largely been replaced by the wide-flange beam.
- A beam of either type is specified by stating its depth in millimeters and its mass per unit length in kilograms per meter. For example, the designation $W610 \times 140$ indicates a wide-flange beam with a nominal depth of 610 mm and a mass per unit length of 140 kg/m.
- When a structural section is selected to be used as a beam, the section modulus must be equal to or greater than the section modulus determined by the flexure equation; that is,

$$S \geq \frac{|M|_{\max}}{\sigma_w}$$

This equation indicates that the section modulus of the selected beam must be equal to or greater than the ratio of the bending moment to the working stress.

- If a beam is very slender (large $\frac{L}{r}$), it may fail by lateral buckling before the working stress is reached.
- Lateral buckling entails loss of resistance resulting from a combination of sideways bending and twisting.
- I-beams are particularly vulnerable to lateral buckling because of their low torsional rigidity and small moment of inertia about the axis parallel to the web.
- When lateral deflection is prevented by a floor system, or by bracing the flanges at proper intervals, the full allowable stresses may be used; otherwise, reduced stresses should be specified in design.
- Formulas for the reduction of the allowable stress are specified by various professional organizations, such as the American Institute of Steel Construction (AISC). In the elementary analysis, we assume that all beams are properly braced against lateral deflection.

Procedure for selecting standard shape

A design engineer is often required to select the lightest standard structural shape (such as a W-shape) that can carry a given loading in addition to the weight of the beam. Following is an outline of the selection process:

- Neglecting the weight of the beam, draw the bending moment diagram to find the largest bending moment $M_{\max}$.
- Determine the minimum allowable section modulus from $S \geq \frac{|M|_{\max}}{\sigma_w}$, where σ_w is the working stress.
- Choose the lightest shape from the list of structural shapes (Standard table) for which $S \geq S_{min}$ and note its weight.
- Calculate the maximum bending stress σ_{max} in the selected beam caused by the prescribed loading plus the *weight of the beam*.
- If $\sigma_{max} \leq \sigma_w$, the selection is finished. Otherwise, the second-lightest shape with $S \geq S_{min}$ must be considered and the maximum bending stress recalculated. The process must be repeated until a satisfactory shape is found.